Properties of Right One-Way Jumping Finite Automata

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Abstract. Right one-way jumping finite automata (ROWJFAs), were recently introduced in [H. Chigahara, S. Z. Fazekas, A. Yamamura: One-Way Jumping Finite Automata, Internat. J. Found. Comput. Sci., 27(3), 2016] and are jumping automata that process the input in a discontinuous way with the restriction that the input head reads deterministically from left-to-right starting from the leftmost letter in the input and when it reaches the end of the input word, it returns to the beginning and continues the computation. We solve most of the open problems of these devices. In particular, we characterize the family of permutation closed languages accepted by ROWJFAs in terms of Myhill-Nerode equivalence classes. Using this, we investigate closure and non-closure properties as well as inclusion relations to other language families. We also give more characterizations of languages accepted by ROWJFAs for some interesting cases.

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1 Introduction

Jumping finite automata [11] are a machine model for discontinuous information processing. Roughly speaking, a jumping finite automaton is an ordinary finite automaton, which is allowed to read letters from anywhere in the input string, not necessarily only from the left of the remaining input. In a series of papers [1, 6, 7, 14] different aspects of jumping finite automata were investigated, such as, e.g., inclusion relations, closure and non-closure results, decision problems, computational complexity of jumping finite automata problems, etc. Shortly after the introduction of jumping automata a variant of this machine model was defined, namely (right) one-way jumping finite automata [3]. There the device moves the input head deterministically from left-to-right starting from the leftmost letter in the input and when it reaches the end of the input word, it returns to the beginning and continues the computation. As in the case of ordinary jumping finite automata inclusion relations to well-known formal language families, closure and non-closure results under standard formal language operations were investigated. Nevertheless, a series of problems on right one-way jumping automata (ROWJFAs) remained open in [3]. This is the starting point of our investigation.

First we develop a characterization of (permutation closed) languages that are accepted by ROWJFAs in terms of the Myhill-Nerode relation. It is shown that the permutation closed language \( L \) belongs to \( \text{ROWJ} \), the family of all languages accepted by ROWJFAs, if and only if \( L \) can be written as the finite union of Myhill-Nerode equivalence classes. Observe, that the overall number of equivalence classes can be infinite. This result nicely contrasts the characterization of regular languages, which requires that the overall number of equivalence classes is finite. The characterization allows us to identify languages that are not accepted by ROWJFAs, which are useful to prove non-closure results on standard formal language operations. In this way we solve all of the open problems from [3] on the inclusion relations of ROWJFAs languages to other language families and on their closure properties. It is shown that the family \( \text{ROWJ} \) is an anti-abstract family of languages (anti-AFL), that is, it is not closed under any of the operations \( \lambda \)-free homomorphism, inverse homomorphism, intersection with regular sets, union, concatenation, or Kleene star. This is a little bit surprising for a language family defined by a deterministic automaton model. Although anti-AFLs are sometimes referred to an “unfortunate family of languages” there is linguistic evidence that such language families might be of crucial importance, since in [4] it was shown that the family of natural languages is an anti-AFL. On the other hand, the family \( \text{pROWJ} \), of all permutation closed languages in \( \text{ROWJ} \), almost form an anti-AFL, since this language family is closed under inverse homomorphism. Moreover, we obtain further characterizations of languages accepted by ROWJFAs. For instance, we show that

1. language \( wL \) is in \( \text{ROWJ} \) if and only if \( L \) is in \( \text{ROWJ} \),
2. language \( Lw \) is in \( \text{ROWJ} \) if and only if \( L \) is regular, and
3. language \( L_1L_2 \) is in \( \text{ROWJ} \) if and only if \( L_1 \) is regular and \( L_2 \) is in \( \text{ROWJ} \), where \( L_1 \) and \( L_2 \) have to fulfil some further easy pre-conditions.

The latter result is in similar vein as a result in [9] on linear context-free languages, where it was shown that \( L_1L_2 \) is a linear context-free language if and only if \( L_1 \) is regular and \( L_2 \) at most linear context free. Finally another characterization is given for letter bounded ROWJFA languages, namely, the language \( L \subseteq a_1^*a_2^*\ldots a_n^* \) is in \( \text{ROWJ} \) if and only if \( L \) is regular. This result nicely generalizes the fact that every unary language accepted by an ROWFA is regular.

The paper is organized as follows: in the next section we introduce the necessary notations on (one-way) jumping finite automata. Then we prove a characterization of the language family \( \text{ROWJ} \) in terms of the Myhill-Nerode equivalence relation in Section 3. Then Section 4
is devoted to inclusion relations between \textbf{ROWJ} and standard language families from formal language theory. There it is shown that the language family \textbf{ROWJ} is incomparable to the family \textbf{JFA}, of all languages accepted by jumping finite automata, solving an open problem from \cite{3}. Closure properties of the language family in question and their permutation closed variant are investigated in Section 5. Finally, in Section 6 more characterizations of languages accepted by \textbf{ROWJFA}s are developed.

2 Preliminaries

We assume the reader to be familiar with the basics in automata and formal language theory as contained, for example, in \cite{10}. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. We use $\subseteq$ for inclusion, and $\subset$ for proper inclusion. Let $\Sigma$ be an alphabet. Then $\Sigma^*$ is the set of all words over $\Sigma$, including the empty word $\lambda$. For a language $L \subseteq \Sigma^*$ define the set $\text{perm}(L) = \bigcup_{w \in L} \text{perm}(w)$, where $\text{perm}(w) = \{ v \in \Sigma^* \mid v$ is a permutation of $w \}$. Then a language $L$ is called permutation closed if $L = \text{perm}(L)$. The length of a word $w \in \Sigma^*$ is denoted by $|w|$. For the number of occurrences of a symbol $a$ in $w$ we use the notation $|w|_a$. We denote the powerset of a set $S$ by $2^S$. For $\Sigma = \{a_1, a_2, \ldots, a_k\}$, the Parikh-mapping $\psi : \Sigma^* \rightarrow \mathbb{N}^k$ is the function $w \mapsto ([|w|_{a_1}], |w|_{a_2}, \ldots, |w|_{a_k})$. A language $L \subseteq \Sigma^*$ is called semilinear if its Parikh-image $\psi(L)$ is a semilinear subset of $\mathbb{N}^k$, a definition of those can be found in \cite{8}.

The elements of $\mathbb{N}^k$ can be partially ordered by the $\leq$-relation on vectors. For $x, y \in \mathbb{N}^k$ we write $x \leq y$ if all components of $x$ are less or equal to the corresponding components of $y$. The value $\|x\|$ is the maximum norm of $x$, that is, $\|x\| = \max\{x_i \mid 1 \leq i \leq k\}$.

Let $\Sigma$ be an alphabet and $v, w \in \Sigma^*$. We say that word $v$ is a prefix of $w$ if there is an $x \in \Sigma^*$ with $w = vx$ and $v$ is a sub-word of $w$ if there are $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n+1} \in \Sigma^*$ with $v = x_1x_2 \cdots x_n$ and $w = y_1x_1y_2x_2 \cdots y_nx_{n+1}$, for some $n \geq 0$. A language $L \subseteq \Sigma^*$ is called prefix-free if and only if there are no words $v, w \in L$ such that $v \neq w$ and $v$ is a prefix of $w$.

For an alphabet $\Sigma$ and a language $L \subseteq \Sigma^*$, let $\sim_L$ be the Myhill-Nerode equivalence relation on $\Sigma^*$. So, for $v, w \in \Sigma^*$, we have $v \sim_L w$ if and only if $vu \in L \iff wu \in L$, for all $u \in \Sigma^*$, holds. For $w \in \Sigma^*$, we call the equivalence class $[w]_{\sim_L}$ positive if and only if $w \in L$. Otherwise, the equivalence class $[w]_{\sim_L}$ is called negative.

A deterministic finite automaton (DFA) is defined as a tuple $A = (Q, \Sigma, R, s, F)$, where $Q$ is the finite set of states, $\Sigma$ is the finite input alphabet, $\Sigma \cap Q = \emptyset$, $R$ is a partial function from $Q \times \Sigma$ to $Q$, $s \in Q$ is the start state, and $F \subseteq Q$ is the set of final states. Elements of $R$ are referred to as rules of $A$ and we write $p \xrightarrow{a} q \in R$ instead of $R(p, a) = q$. A configuration of $A$ is a string in $Q \Sigma^*$. A DFA makes a transition from configuration $paw$ to configuration $qw$ if $pa \xrightarrow{a} q \in R$, where $p, q \in Q$, $a \in \Sigma$, and $w \in \Sigma^*$. We denote this by $paw \xrightarrow{a} qw$ or just $paw \xrightarrow{a} qw$ if it is clear which DFA we are referring to. In the standard manner, we extend $\xrightarrow{a}$ to $\xrightarrow{a}^n$, where $n \geq 0$. Let $+\xrightarrow{a}$ and $\xrightarrow{a}^+$ denote the transitive closure of $\xrightarrow{a}$ and the transitive-reflexive closure of $\xrightarrow{a}$, respectively.

The language accepted by $A$ is $L(A) = \{ w \in \Sigma^* \mid \exists f \in F : sw \xrightarrow{a}^+ f \}$. We say that $A$ accepts $w \in \Sigma^*$ if $w \in L(A)$ and that $A$ rejects $w$ otherwise. The family of languages accepted by DFAs is referred to as $\text{REG}$.

A jumping finite automaton (JFA) is a tuple $A = (Q, \Sigma, R, s, F)$, where $Q$, $\Sigma$, $R$, $s$, and $F$ are the same as in the case of DFAs. A configuration of $A$ is a string in $Q \Sigma^* Q^*$. The binary jumping relation, symbolically denoted by $\wedge_A$, over $Q^* Q^*$ is defined as follows. Let $x, z, x', z'$ be strings in $\Sigma^*$ such that $xz = x'z'$ and $py \xrightarrow{a} q \in R$. Then, the automaton $A$ makes a jump from $xpyz$ to $x'qz'$, symbolically written as $xpyz \wedge_A x'qz'$ or just $xpyz \wedge x'qz'$ if it is clear which JFA we are referring to. In the standard manner, we extend $\wedge$ to $\wedge^n$, where $n \geq 0$.
Let $\odot^+$ and $\odot^*$ denote the transitive closure of $\odot$ and the transitive-reflexive closure of $\odot$, respectively. Then, the language accepted by $A$ is $L(A) = \{ w \mid u, v \in \Sigma^*, \exists f \in F : usv \odot^* f \}$. We say that $A$ accepts $w \in \Sigma^*$ if $w \in L(A)$ and that $A$ rejects $w$ otherwise. Let $\text{JFA}$ be the family of all languages that are accepted by JFAs.

A right one-way jumping finite automaton (ROWJFA) is a tuple $A = (Q, \Sigma, R, s, F)$, where the elements $Q$, $\Sigma$, $R$, $s$, and $F$ are defined as in a DFA. A configuration of $A$ is a string in $Q \Sigma^*$. We simply write $\rtimes \Delta_{\text{JFA}}$ to the configuration $qyx$, symbolically written as $pxay \rtimes A qyx$. We also write $pxay \rtimes qyx$ if it is clear which ROWJFA we are referring to. In the standard manner, we extend $\odot$ to $\odot^n$, where $n \geq 0$. Let $\odot^+$ and $\odot^*$ denote the transitive closure of $\odot$ and the transitive-reflexive closure of $\odot$, respectively. The language accepted by $A$ is the set $L(A) = \{ w \in \Sigma^* \mid \exists f \in F : sw \odot^* f \}$. We say that $A$ accepts $w \in \Sigma^*$ if $w \in L(A)$ and that $A$ rejects $w$ otherwise. Let $\text{ROWJ}$ be the family of all languages that are accepted by ROWJFAs. Furthermore, for $n \geq 0$, be the class of all languages accepted by ROWJFAs with at most $n$ accepting states is referred to as $\text{ROWJ}_n$.

Besides the above mentioned language families let $\text{FIN}$, $\text{DCF}$, $\text{CF}$, and $\text{CS}$ be the families of finite, deterministic context-free, context-free, and context-sensitive languages. Moreover, we are interested in permutation closed language families. These language families are referred to by a prefix $p$. E.g., $p\text{ROWJ}$ denotes the language family of all permutation closed $\text{ROWJ}$ languages.

Sometimes, for a DFA $A$, we will also consider the relations $\odot$ and $\odot$, that we get by interpreting $A$ as a JFA or a ROWJFA. The following three languages are associated to $A$:

- $L_D(A)$ is the language accepted by $A$, interpreted as an ordinary DFA.
- $L_J(A)$ is the language accepted by $A$, interpreted as an JFA.
- $L_R(A)$ is the language accepted by $A$, interpreted as an ROWJFA.

From a result in [12] and from [3, Theorem 10], we get

$$L_D(A) \subseteq L_R(A) \subseteq L_J(A) = \text{perm}(L_D(A)). \quad (1)$$

As a consequence, we have $\text{JFA} = p\text{JFA}$. We give an example of a DFA $A$ with the property that $L_D(A) \subset L_R(A) \subset L_J(A)$:

**Example 1.** Let $A$ be the DFA

$$A = (\{q_0, q_1, q_2, q_3\}, \{a, b\}, \Sigma, q_0, \{q_3\}),$$

where $R$ consists of the rules $q_0b \rightarrow q_1$, $q_0a \rightarrow q_2$, $q_2b \rightarrow q_3$, and $q_3a \rightarrow q_2$. The automaton $A$ is depicted in Figure 1.

It holds $L_D(A) = (ab)^+$ and

$$L_J(A) = \text{perm}((ab)^+) = \{ w \in \{a, b\}^+ \mid |w|_a = |w|_b \}.$$  

Then again, it is not hard to see that $L_R(A) = \{ w \in \{a, b\}^* \mid |w|_a = |w|_b \}$. Notice that this language is non-regular and not closed under permutation.
Let \( \text{Lemma 2.} \) Let \( A = (Q, \Sigma, R, s, F) \) be a DFA. Consider two words \( v, w \in \Sigma^* \), states \( p, q \in Q \), and an \( n \geq 0 \) with \( pv \circ^n qw \). Then, there is a word \( x \in \Sigma^* \) such that \( xw \) is a permutation of \( v \), and \( px \vdash^n q \).

**Proof.** We prove this by induction on \( n \). If \( n = 0 \), we have \( pv = qw \) and just set \( x = \lambda \). Now, assume \( n > 0 \) and that the lemma is true for the relation \( \circ^{n-1} \). We get a state \( r \in Q \), a symbol \( a \in \Sigma_r \), and words \( y \in (\Sigma \setminus \Sigma_r)^* \) and \( z \in \Sigma^* \) such that \( w = zy \) and \( pv \circ^{n-1} ryaz \circ qw \).

By the induction hypothesis, there is a word \( x' \in \Sigma^* \) such that \( x'yaz \) is a permutation of \( v \), and \( px' \vdash^{n-1} r \). Set \( x = x'a \). Then, the word \( xw = x'azy \) is a permutation of \( x'yaz \), which is a permutation of \( v \). Furthermore, we get \( px = px'a \vdash^{n-1} ra \vdash q \). This proves the lemma. \( \square \)

**3 A Characterization of Permutation Closed Languages Accepted by ROWJFAs**

By the Myhill-Nerode theorem, a language \( L \) is regular if and only if the Myhill-Nerode equivalence relation \( \sim_L \) has only a finite number of equivalence classes. Moreover, the number of equivalence classes equals the number of states of the minimal DFA accepting \( L \), see for example [10]. We can give a similar characterization for permutation closed languages that are accepted by an ROWJFA.

**Theorem 3.** Let \( L \) be a permutation closed language and \( n \geq 0 \). Then, the language \( L \) is in \( \text{ROWJ}_n \) if and only if the Myhill-Nerode equivalence relation \( \sim_L \) has at most \( n \) positive equivalence classes.

**Proof.** First, assume that \( L \) is in \( \text{ROWJ}_n \) and let \( A = (Q, \Sigma, R, s, F) \) be a DFA with \( |F| \leq n \) and \( L_R(A) = L \). Consider \( v, w \in L \) and \( f \in F \) with \( sv \circ^* f \) and \( sw \circ^* f \). Lemma 2 shows that there are permutations \( v' \) and \( w' \) of \( v \) and \( w \) with \( sv' \vdash^* f \) and \( sw' \vdash^* f \). Because language \( L \) is closed under permutation we have \( v \sim_L v' \) and \( w \sim_L w' \). Now, let \( u \in \Sigma^* \). Thus \( sv'u \circ^* fu \) and \( sw'u \circ^* fu \). That gives us \( w'u \in L \iff (\exists g \in F : fu \circ^* g) \iff w'u \in L \).

We have shown \( v \sim_L v' \sim_L w' \sim_L w \). From \( L = \bigcup_{f \in F} \{ w \in \Sigma^* | sv \circ^* f \} \), we conclude that \( |L/ \sim_L | \leq |F| \leq n \), which means that \( \sim_L \) has at most \( n \) positive equivalence classes.

Assume now that \( \sim_L \) has at most \( n \) positive equivalence classes and let \( \Sigma = \{a_1, a_2, \ldots, a_k\} \) be an alphabet with \( L \subseteq \Sigma^* \). Set \( L_\lambda = L \cup \{\lambda\} \). Define the map \( S : L_\lambda / \sim_L \rightarrow 2^{\mathbb{N}^k} \) through \( [w] \rightarrow \{ x \in \mathbb{N}^k \setminus \mathbf{0} | \psi^{-1}(\psi(w) + x) \subseteq L \} \).

The definition of \( \sim_L \) and the fact that \( L \) is closed under permutation make the map \( S \) well-defined. Consider the relation \( \leq \) on \( \mathbb{N}^k \). For each \( [w] \in L_\lambda / \sim_L \), let \( M([w]) \) be the set of minimal
elements of $S([w])$. So, for every $[w] \in L_\lambda \sim_L$ and $x \in S([w])$, there is an $x_0 \in M([w])$ such that $x_0 \leq x$. Due to [5] each subset of $\mathbb{N}^k$ has only a finite number of minimal elements, so the sets $M([w])$ are finite. For $i \in \{1, 2, \ldots, k\}$, let $\pi_i : \mathbb{N}^k \to \mathbb{N}$ be the canonical projection on the $i$th factor and set

$$m_i = \max \left( \bigcup_{[w] \in L_\lambda \sim_L} \{ \pi_i(x) \mid x \in M([w]) \} \right),$$

where $\max(\emptyset)$ should be 0. We have $m_i < \infty$, for all $i \in \{1, 2, \ldots, k\}$, because of $|L_\lambda \sim_L| \leq n+1$. Let

$$Q = \left\{ q_{[w] \sim_L} \mid w \in L_\lambda, v \in \Sigma^* \text{ with } |v|_{a_i} \leq m_i, \text{ for all } i \in \{1, 2, \ldots, k\} \right\}$$

be a set of states. The finiteness of $L_\lambda \sim_L$ implies that $Q$ is also finite. Set

$$F = \left\{ q_{[w] \sim_L} \mid w \in L \right\} \subseteq Q.$$

We get $|F| = |L_\lambda \sim_L| \leq n$. Define the partial mapping $R : Q \times \Sigma \to Q$ by $R(q_{[y] \sim_L}, a) = q_{[ya] \sim_L}$ if $q_{[ya] \sim_L} \in Q$, and $R(q_{[y] \sim_L}, a)$ be undefined otherwise, for $a \in \Sigma$ and $y \in \Sigma^*$ with $q_{[y] \sim_L} \in Q$. Consider the DFA $A = (Q, \Sigma, R, q_{\lambda L}, F)$. We will show that $L_R(A) = L$.

First, let $y \in L_R(A)$. Then, there exists $w \in L$ with $q_{[\lambda L} y \circ^* q_{[w] \sim_L}$. From Lemma 2 it follows that there is a permutation $y'$ of $y$ with $q_{[\lambda L} y' \circ^* q_{[w] \sim_L}$. Now, the definition of $R$ tells us $y' \sim_L w$. We get $y' \in L$ and also $y \in L$, because $L$ is closed under permutation. That shows the inclusion $L_R(A) \subseteq L$.

Now, let $y \in \Sigma^* \setminus L_R(A)$. There are two possibilities:

1. There is $w \in \Sigma^* \setminus L$ with $q_{[w] \sim_L} \in Q$ such that $q_{[\lambda L} y \circ^* q_{[w] \sim_L}$. Then, there is a permutation $y'$ of $y$ with $q_{[\lambda L} y' \circ^* q_{[w] \sim_L}$. We get $y' \sim_L w$. It follows $y' \notin L$, which gives us $y \notin L$.

2. There are $w \in L_\lambda$, a $v \in \Sigma^*$ with $|v|_{a_i} \leq m_i$, for all $i \in \{1, 2, \ldots, k\}$, and a word $z$ with $z \in (\Sigma \setminus \Sigma_{q_{[w] \sim_L}})^+$ such that $q_{[\lambda L} y \circ^* q_{[w] \sim_L} z$. By Lemma 2 there is a $y' \in \Sigma^*$ such that $y'z$ is a permutation of $y$ and satisfies $q_{[\lambda L} y' \circ^* q_{[w] \sim_L}$. We get $y' \sim_L uvw$. Set

$$U = \bigcup_{t \in \Sigma^*} \{ u \in \Sigma^* \mid ut \in \text{perm}(v) \text{ and } wu \in L_\lambda \}.$$

We have $\lambda \in U$. Let $u_0 \in U$ such that $|u_0| = \max(\{|u| \mid u \in U\})$ and let $t_0 \in \Sigma^*$ such that $u_0 t_0 \in \text{perm}(v)$. It follows that $|t_0|_{a_i} \leq |v|_{a_i} \leq m_i$, for all $i \in \{1, 2, \ldots, k\}$, and that there exists no $x \in M([wu_0] \sim_L)$ with $x \leq \psi(t_0)$. Otherwise, we would have an $x' \in \psi^{-1}(x)$ which is a non-empty sub-word of $t_0$ such that $w_0 x' \in L_\lambda$, which implies $u_0 x' \notin U$. However, this is a contradiction to the maximality of $|u_0|$. That shows that there is no $x \in M([wu_0] \sim_L)$ with $x \leq \psi(t_0)$. Let now $x_0 \in M([wu_0] \sim_L)$. Then $|t_0|_{a_j} < \pi_j(x_0)$ for some element $j \in \{1, 2, \ldots, k\}$. Because of $|t_0|_{a_i} \leq m_i$, for all $i$ with $i \in \{1, 2, \ldots, k\}$, and since we have the equality $z \in (\Sigma \setminus \Sigma_{q_{[w] \sim_L}})^+$, we get $|z|_{a_j} = 0$. That gives $|t_0 z|_{a_j} < \pi_j(x_0)$ and that $\psi(t_0 z) \geq x_0$ is false. So, we have shown $\psi(t_0 z) \notin S([wu_0] \sim_L)$, which immediately implies $w_0 t_0 z \notin L$. From $w_0 t_0 z \sim_L w v z \sim_L y' z \sim_L y$, it follows that $y \notin L$.

We have seen $L_R(A) = L$. This shows that $L$ is in ROWJ_n. □
The previous theorem allows us to determine for a lot of interesting languages whether they belong to \( \text{ROWJ} \) or not.

**Corollary 4.** Let \( L \) be a permutation closed language. Then, the language \( L \) is in \( \text{ROWJ} \) if and only if the Myhill-Nerode equivalence relation \( \sim_L \) has only a finite number of positive equivalence classes.

An application of the last corollary is the following.

**Lemma 5.** The language \( L = \{ w \in \{a, b\}^* \mid |w|_b = 0 \lor |w|_b = |w|_a \} \) is not included in \( \text{ROWJ} \).

**Proof.** Obviously, the language \( L \) is closed under permutation. For \( \sim_L \), the positive equivalence classes \( [a^0], [a^1], \ldots \) are pairwise different, since \( a^n b^m \in L \) if and only if \( m \in \{0, n\} \). Corollary 4 tells us that \( L \) is not in \( \text{ROWJ} \).

There are counterexamples for both implications of Corollary 4, if we do not assume that the language \( L \) is closed under permutation. For instance, set \( L = \{ a^n b^n \mid n \geq 0 \} \), which was shown to be not in \( \text{ROWJ} \) in [3]. Then, the positive equivalence classes of \( \sim_L \) are \( [\lambda] \) and \( [ab] \).

On the other hand, we have:

**Lemma 6.** There is a language \( L \) in \( \text{ROWJ} \) such that \( \sim_L \) has an infinite number of positive equivalence classes.

**Proof.** Let \( A \) be the ROWJFA

\[
(\{q_0, q_1, q_2, q_3, q_4\}, \{a, b\}, R, q_0, \{q_2, q_3\}),
\]

where \( R \) consists of the rules

\[
q_0 b \to q_1, \quad q_1 a \to q_2, \quad q_2 a \to q_2, \quad q_1 b \to q_3, \quad q_3 a \to q_4, \quad \text{and} \quad q_4 b \to q_3.
\]

The ROWJFA \( A \) is depicted in Figure 2. Let \( n > 0 \). Then, we have \( q_0 a^n b \cup q_1 a^n \cup^+ q_2 \), which gives \( a^n b \in L(A) \). We also have

\[
q_0 a^n b b^n \cup q_1 b^n a^n \cup q_3 b^n a^n \cup q_3 b^{n-1} a^{n-1} \cup^+ \ldots \cup^+ q_3 b^0 a^0.
\]

It follows \( a^n b b^n \in L(A) \). Whenever \( A \) is in state \( q_3 \), the number of read \( b \)'s equals the number of read \( a \)'s plus 2. That implies \( a^n b b^m \notin L(A) \), for all \( m \geq 0 \) with \( m \neq n \). So, the positive equivalence classes \( [a^1 b]_{\sim_L(A)}, [a^2 b]_{\sim_L(A)}, \ldots \) are pairwise different. This proves the lemma.

From Corollary 4 we conclude the following equivalence.

**Corollary 7.** Let \( L \) be a permutation closed \( \text{ROWJ} \) language over the alphabet \( \Sigma \). Then, the language \( L \) is regular if and only if \( \Sigma^* \setminus L \) is in \( \text{ROWJ} \).
Proof. By Corollary 4, the Myhill-Nerode equivalence relation $\sim_L$ has only a finite number of positive equivalence classes. So, $L$ is regular if and only if $\sim_L$ has only a finite number of negative equivalence classes, by the Myhill-Nerode theorem. The latter condition holds if and only if $\sim_{\Sigma \setminus L}$ has only a finite number of positive equivalence classes. Again by Corollary 4, this is equivalent to the condition that $\Sigma^* \setminus L$ is in ROWJ, because the complement of a permutation closed language is also permutation closed. \qed

The previous corollary gives us:

Lemma 8. The language \( \{ w \in \{a, b\}^* \mid |w|_a \neq |w|_b \} \) is not in ROWJ.

Proof. Consider the permutation closed non-regular language \( L = \{ w \in \{a, b\}^* \mid |w|_a = |w|_b \} \) over the alphabet $\Sigma = \{a, b\}$. In [3], it was shown that $L$ is in ROWJ. Now, by Corollary 7, the language

\[ \Sigma^* \setminus L = \{ w \in \{a, b\}^* \mid |w|_a \neq |w|_b \} \]

is not in ROWJ. \qed

Having the statement of Theorem 3, it is natural to ask, which numbers arise as the number of positive equivalence classes of the Myhill-Nerode equivalence relation $\sim_L$ of a permutation closed language $L$. The answer is, that all natural numbers arise this way, even if we restrict ourselves to some special families:

Theorem 9. For each $n > 0$, there is a permutation closed language which is (1) finite, (2) regular, but infinite, (3) context-free, but non-regular, (4) non-context-free such that the corresponding Myhill-Nerode equivalence relation has exactly $n$ positive equivalence classes.

Proof. For each $n > 0$, set

\[
L_n = \{ a^m \mid m < n \}, \\
M_n = \{ a^m \mid m \mod (n + 1) \neq n \}, \\
N_n = \{ w \in \{a, b\}^+ \mid |w|_a = |w|_b \} \cup \{ c^m \mid 0 < m < n \}, \\
O_n = \{ w \in \{a, b, c\}^+ \mid |w|_a = |w|_b = |w|_c \} \cup \{ d^m \mid 0 < m < n \}.
\]

All these languages are closed under permutation. Obviously, the language $L_n$ is finite. The positive equivalence classes of $\sim_{L_n}$ are $[a^0], [a^1], \ldots, [a^{n-1}]$.

While $M_n$ is infinite, the equivalence classes of $\sim_{M_n}$ are $[a^0], [a^1], \ldots, [a^n]$. So, the language $M_n$ is regular, by the Myhill-Nerode Theorem. Only the last mentioned equivalence class is negative. Therefore, there are exactly $n$ positive equivalence classes of $\sim_{M_n}$.

For $\sim_{N_n}$, the equivalence classes $[a^0], [a^1], \ldots$ are pairwise different. So, the language $N_n$ is non-regular, by the Myhill-Nerode Theorem. It is context-free, because it is the union of a well known context-free language and a finite language. The positive equivalence classes of the relation $\sim_{N_n}$ are $[ab], [c^1], [c^2], \ldots, [c^{n-1}]$.

If $O_n$ was context-free, then

\[ O_n \cap \{a, b, c\}^+ = \{ w \in \{a, b, c\}^+ \mid |w|_a = |w|_b = |w|_c \} \]

would also be context-free, as the intersection of a context-free and a regular language. However, the language $\{ w \in \{a, b, c\}^+ \mid |w|_a = |w|_b = |w|_c \}$ is a well known non-context-free language. It follows, that $O_n$ is also non-context-free. The positive equivalence classes of the relation $\sim_{O_n}$ are $[abc], [d^1], [d^2], \ldots, [d^{n-1}]$. This proves the theorem. \qed

8
The previous theorem, together with Theorem 3, implies that the language families $ROWJ_n$ form a proper hierarchy, even if we only consider languages out of special language families:

**Corollary 10.** For all $n \geq 0$, we have

\[
ROWJ_n \subset ROWJ_{n+1}, \\
ROWJ_n \cap \text{FIN} \subset ROWJ_{n+1} \cap \text{FIN}, \\
ROWJ_n \cap (\text{REG} \setminus \text{FIN}) \subset ROWJ_{n+1} \cap (\text{REG} \setminus \text{FIN}), \\
ROWJ_n \cap (\text{CF} \setminus \text{REG}) \subset ROWJ_{n+1} \cap (\text{CF} \setminus \text{REG}), \\
ROWJ_n \cap (\text{CS} \setminus \text{CF}) \subset ROWJ_{n+1} \cap (\text{CS} \setminus \text{CF}).
\]

The statement remains valid if restricted to permutation closed languages.

### 4 Inclusion Relations Between Language Families

We investigate inclusion relations between $ROWJ$ and other important languages families. The following inclusion relations were given in [3]:

- $\text{REG} \subset ROWJ$,
- $ROWJ$ and $\text{CF}$ are incomparable,
- $ROWJ \not\subset JFA$.

It was stated as an open problem if $JFA \subset ROWJ$. We can answer this:

**Theorem 11.** The language families $ROWJ$ and $JFA$ are incomparable.

**Proof.** The language $\{ w \in \{a,b\}^* \mid |w|_b = 0 \lor |w|_b = |w|_a \}$ is not included in the family $ROWJ$, by Lemma 5, but it belongs to $JFA$, because it is the permutation closure of the regular language $a^* \cup (ab)^*$. So, we get $JFA \not\subset ROWJ$. Together with the result $ROWJ \not\subset JFA$ from [3] the incomparability of the language families $ROWJ$ and $JFA$ follows.

For the complexity of $ROWJ$, we get:

**Theorem 12.** The language family $ROWJ$ is included in $\text{DTIME}(n^2)$ and $\text{DSPACE}(n)$.

**Proof.** Right revolving automata were described in [2]. It was shown that every language accepted by a deterministic right revolving automaton belongs to both classes $\text{DTIME}(n^2)$ and $\text{DSPACE}(n)$. In [3] it was proven that $ROWJ$ is properly included in the family of languages accepted by deterministic right revolving automata.

This implies that $ROWJ$ is properly included in $\text{CS}$:

**Theorem 13.** We have $ROWJ \subset \text{CS}$.

**Proof.** From Theorem 12 we get $ROWJ \subseteq \text{CS}$. On the other hand, $ROWJ$ and $\text{CF}$ are incomparable, which proves the theorem.

We also get a result for the inclusion relation between $ROWJ$ and the family of deterministic context-free languages:

**Theorem 14.** The language families $ROWJ$ and $DCF$ are incomparable.

**Proof.** The families $ROWJ$ and $\text{CF}$ are incomparable, so there are non context-free languages in $ROWJ$. Moreover, it was shown in [3] that the deterministic context-free language

\[
\{ a^n b^n \mid n \geq 0 \}
\]

is not accepted by any $ROWJFA$.
By the famous result in [13], every context-free language is semilinear. In [3] it was proven that every language in JFA is also semilinear. This holds for ROWJ, too:

**Theorem 15.** Every language in ROWJ is semilinear.

**Proof.** For every language \( L \) in ROWJ, there exists a DFA \( A \) such that \( L = L_R(A) \). From (1) we get

\[
\psi(L_D(A)) \subseteq \psi(L_R(A)) \subseteq \psi(L_J(A)).
\]

Because of \( L_J(A) = \text{perm}(L_D(A)) \), we have \( \psi(L_J(A)) = \psi(L_D(A)) \). So,

\[
\psi(L) = \psi(L_R(A)) = \psi(L_D(A)),
\]

which is a semilinear set, because \( L_D(A) \) is regular. \( \square \)

We now consider inclusion relations between families of permutation closed languages. It holds

\[
pFIN \subset pREG \subset pDCF \subseteq pCF \subset pCS,
\]

(2)

witness languages are \( a^* \), \( \{ w \in \{a,b\}^* \mid |w|_a = |w|_b \} \), and \( \{ a^{2n} \mid n \geq 0 \} \). There is also a language that distinguishes pDCF and pCF:

**Theorem 16.** We have pDCF \( \subset \) pCF.

**Proof.** Consider the permutation closed language

\[
L = \{ w \in \{a,b,c\}^* \mid |w|_a = |w|_b \lor |w|_b = |w|_c \}.
\]

It is context-free as the union of two context-free languages. If \( L \) was deterministic context-free, then

\[
L' = L \cap a^*b^*c^* = \left\{ a^i b^j c^k \mid (i,j,k \geq 0) \land (i = j \lor j = k) \right\}
\]

was also deterministic context-free as the intersection of a deterministic context-free and a regular language. However, language \( L' \) is not deterministic context-free, as shown in [10]. Hence language \( L \) is also not deterministic context-free, which proves the theorem. \( \square \)

The next theorem places JFA in the hierarchy (2).

**Theorem 17.** We have pCF \( \subset \) JFA \( \subset \) pCS.

**Proof.** The first strict inclusion is seen as follows: it was shown that every context-free language is semilinear in [13], while in [3] it was proven that JFA is the family of all permutation closed semilinear languages. So, we get pCF \( \subseteq \) JFA. On the other hand, the non-context free language

\[
\{ w \in \{a,b,c\}^* \mid |w|_a = |w|_b = |w|_c \}
\]

is in JFA, which was shown in [11]. This proves the first inclusion.

For the second strict inclusion we argue as follows: in [11] it was proven that JFA \( \subset \) CS and that all languages in JFA are closed under permutation. This gives us JFA \( \subseteq \) pCS. The permutation closed context-sensitive language \( \{ a^{2n} \mid n \geq 0 \} \) is not in JFA, because it is not semilinear. \( \square \)
So, we get
\[ p_{FIN} \subseteq p_{REG} \subseteq p_{DCF} \subseteq p_{CF} \subseteq J_{FA} \subseteq p_{CS}. \]

We investigate the inclusion relations of \( p_{ROWJ} \), now.

**Theorem 18.** We have \( p_{REG} \subseteq p_{ROWJ} \subseteq J_{FA} \).

*Proof.* Since \( REG \subseteq ROWJ \), we have \( p_{REG} \subseteq p_{ROWJ} \). The permutation closed, non-regular language \( \{ w \in \{a, b\}^* \mid |w|_a = |w|_b \} \) was shown to be included in \( ROWJ \) in [3].

Theorem 15 implies \( p_{ROWJ} \subseteq J_{FA} \), because \( J_{FA} \) is the family of all permutation closed semilinear languages. On the other hand, Theorem 11 tells us that there is a language in \( J_{FA} \), which is not in \( p_{ROWJ} \). \( \square \)

Next, we consider the inclusion relations between \( p_{ROWJ} \) and the language families \( p_{DCF} \) and \( p_{CF} \).

**Theorem 19.** The language family \( p_{ROWJ} \) is incomparable to \( p_{DCF} \) and to \( p_{CF} \).

*Proof.* From [3] we know that the permutation closed, non-context-free language
\[ \{ w \in \{a, b, c\}^* \mid |w|_a = |w|_b = |w|_c \} \]
is in \( ROWJ \). The language \( L = \{ w \in \{a, b\}^* \mid |w|_a = |w|_b \} \) is in \( p_{DCF} \) and so is \( \{a, b\}^* \setminus L \), because \( p_{DCF} \) is closed under complementation. Lemma 8 gives us that \( \{a, b\}^* \setminus L \) is not in \( ROWJ \), which proves the theorem. \( \square \)

Finally, we get from Theorems 11 and 18:

**Theorem 20.** We have \( p_{ROWJ} \subseteq ROWJ \).

## 5 Closure Properties of ROWJ and pROWJ

We consider closure properties of the language families \( ROWJ \) and \( p_{ROWJ} \). Our results are summarized in Table 1.

The language family \( ROWJ \) is not closed under the operations of intersection, intersection with regular languages, reversal, concatenation, concatenation with regular languages from the right, Kleene star, Kleene plus, and substitution. All these properties were proven in [3]. In the following we will show that \( ROWJ \) is also not closed under the operations of union, union with regular languages, complement, concatenation with regular languages from the left, homomorphism, \( \lambda \)-free homomorphism, inverse homomorphism and permutation closure. However, we will prove one positive closure result: the family \( ROWJ \) is closed under concatenation with prefix-free regular languages from the left.

**Theorem 21.** The family \( ROWJ \) is not closed under union and under union with regular languages.

*Proof.* Consider \( L_1 = a^* \) and \( L_2 = \{ w \in \{a, b\}^* \mid |w|_a = |w|_b \} \). The language \( L_1 \) is in \( ROWJ \), because it is regular, while \( L_2 \) was shown to be in \( ROWJ \) in [3]. In Lemma 5 it was shown that the union \( L_1 \cup L_2 \) is not in \( ROWJ \). \( \square \)

Next we consider the complementation operation.

**Theorem 22.** The family \( ROWJ \) is not closed under complement.

*Proof.* While \( \{ w \in \{a, b\}^* \mid |w|_a = |w|_b \} \) is in \( ROWJ \), its complement is not, which was shown in Lemma 8. \( \square \)
From [3] we know that ROWJ is not closed under concatenation, not even under concatenation with regular languages from the right. Also, under concatenation with regular languages from the left, the family ROWJ is not closed:

**Theorem 23.** The family ROWJ is not closed under concatenation with regular languages from the left.

**Proof.** Consider the regular language $L_1 = a^*$ and the ROWJ language

$$L_2 = \{ w \in \{a,b\}^* \mid |w|_a = |w|_b \}.$$

Assume that there is a DFA $A = (Q, \{a,b\}, R, s, F)$ with $L_R(A) = L_1 L_2$. For each $n \geq 0$, there is exactly one $q_n \in F$ with $sa^n \vdash q_n$. Because of $|F| \leq \infty$, there are $0 \leq n < m$ with $q_n = q_m$. Since the word $\lambda a^m b^m$ belongs to $L_1 L_2$, there exists $q \in F$ with $sa^m b^m \vdash q m b^m \vdash q$. This implies that $sa b^m \vdash q m b^m \vdash q$, which gives us $a^m b^m \in L_1 L_2$. That is a contradiction, because of $m > n$. Thus, the language $L_1 L_2$ is not in ROWJ.

If we add the condition that the regular language has to be prefix-free, we get a positive closure result:

**Theorem 24.** The family ROWJ is closed under concatenation with prefix-free regular languages from the left.

**Proof.** For an alphabet $\Sigma$, let $L_1 \subseteq \Sigma^*$ be a prefix-free regular language and moreover $L_2 \subseteq \Sigma^*$ be a ROWJ language. If $\lambda \in L_1$, we have $L_1 = \{ \lambda \}$ and therefore $L_1 L_2 = L_2$. Thus, assume form now on that $\lambda \notin L_1$. Let $A_1 = (Q_1, \Sigma, R_1, s_1, F_1)$ be a DFA with total transition function $R_1$

### Table 1. Closure properties of ROWJ and pROWJ. The gray shaded results are proven in this paper. The non-shaded closure properties for REG are folklore. For ROWJ the closure/non-closure results can be found in [3] and that for the language family JFA in [1, 6, 7, 12].

<table>
<thead>
<tr>
<th>Closed under</th>
<th>Language family</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>REG</td>
</tr>
<tr>
<td>Union</td>
<td>yes</td>
</tr>
<tr>
<td>Union with reg. lang.</td>
<td>yes</td>
</tr>
<tr>
<td>Intersection</td>
<td>yes</td>
</tr>
<tr>
<td>Intersection with reg. lang.</td>
<td>yes</td>
</tr>
<tr>
<td>Complementation</td>
<td>yes</td>
</tr>
<tr>
<td>Reversal</td>
<td>yes</td>
</tr>
<tr>
<td>Concatenation</td>
<td>yes</td>
</tr>
<tr>
<td>Right conc. with reg. lang.</td>
<td>yes</td>
</tr>
<tr>
<td>Left conc. with reg. lang.</td>
<td>yes</td>
</tr>
<tr>
<td>Left conc. with prefix-free reg. lang.</td>
<td>yes</td>
</tr>
<tr>
<td>Kleene star</td>
<td>yes</td>
</tr>
<tr>
<td>Kleene plus</td>
<td>yes</td>
</tr>
<tr>
<td>Homomorphism</td>
<td>yes</td>
</tr>
<tr>
<td>$\lambda$-free homomorphism</td>
<td>yes</td>
</tr>
<tr>
<td>Inv. homomorphism</td>
<td>yes</td>
</tr>
<tr>
<td>Substitution</td>
<td>yes</td>
</tr>
<tr>
<td>Permutation</td>
<td>no</td>
</tr>
</tbody>
</table>
and \( L_D(A_1) = L_1 \). Moreover, let \( A_2 = (Q_2, \Sigma, R_2, s_2, F_2) \) be a DFA with \( L_R(A_2) = L_2 \) and assume \( Q_1 \cap Q_2 = \emptyset \) without loss of generality. Consider the DFA

\[
B = ((Q_1 \setminus F_1) \cup Q_2, \Sigma, S, s_1, F_2),
\]

where \( S \) is defined as follows: for \( (q, a) \in (Q_1 \setminus F_1) \times \Sigma, \) let \( S(q, a) = R_1(q, a) \), if \( R_1(q, a) \notin F_1 \), and \( S(q, a) = s_2 \), otherwise. For \( (q, a) \in Q_2 \times \Sigma, \) the value \( S(q, a) \) is defined if and only if \( R_2(q, a) \) is defined. In this case we have \( S(q, a) = R_2(q, a) \). We will show that \( L_R(B) = L_1 L_2 \).

First, let \( v \in L_1 \) and \( w \in L_2 \). So, there is a symbol \( a \in \Sigma \) and states \( p \in Q_1, q \in F_1 \), and \( r \in F_2 \) such that \( s_1 v \xrightarrow{a} p \) \( a \xrightarrow{p} q \) and \( s_2 w \xrightarrow{a} r \). Because \( L_1 \) is prefix-free, there are no word \( x \in \Sigma^+ \) and \( q' \in F_1 \) such that \( s_1 v \xrightarrow{a} q' x \). This gives us \( s_1 v w \xrightarrow{a} p a w \xrightarrow{a} s_2 w \xrightarrow{a} r \), which implies \( vw \in L_R(B) \).

Let now \( v \in L_R(B) \). Since \( R_1 \) is a total function, there are a symbol \( a \in \Sigma \), words \( w, x \in \Sigma^* \), and states \( p \in Q_1 \setminus F_1 \) and \( q \in F_2 \) such that \( v = w x \) and \( s_1 w a x \xrightarrow{a} p a x \xrightarrow{a} s_2 x \xrightarrow{a} q \). So, there is an \( r \in F_1 \) with \( s_1 w a x \xrightarrow{a} p a x \xrightarrow{a} r \) and \( s_2 x \xrightarrow{a} q \). This gives us \( w a \in L_1 \) and \( x \in L_2 \), which proves the theorem.

The previous theorem allows us for a large family of languages to show that they belong to \( \text{ROWJ} \). From Corollary 4 and Theorem 24 it follows that:

**Corollary 25.** Let \( \Sigma \) be an alphabet and \( w \in \Sigma^* \). Furthermore, let \( L \subseteq \Sigma^* \) be a permutation closed language such that the Myhill-Nerode equivalence relation \( \sim_L \) has only a finite number of positive equivalence classes. Then, the language \( w L \) is in \( \text{ROWJ} \).

For marked concatenation we find a similar result, which can be deduced from Corollary 4 and Theorem 24, too, because \( L_1 a \) is a prefix-free regular language.

**Corollary 26.** Let \( \Sigma \) be an alphabet and \( a \in \Sigma \). Moreover, let \( L_1 \subseteq (\Sigma \setminus \{a\})^* \) be a regular language and \( L_2 \subseteq \Sigma^* \) be a permutation closed language such that the Myhill-Nerode equivalence relation \( \sim_{L_2} \) has only a finite number of positive equivalence classes. Then, the language \( L_1 a L_2 \) is in \( \text{ROWJ} \).

Now, we turn back to the closure properties of \( \text{ROWJ} \).

**Theorem 27.** The family \( \text{ROWJ} \) is not closed under \( \lambda \)-free homomorphism nor under homomorphism.

**Proof.** Consider the permutation closed language

\[
L = a^* \cup \{ w \in \{b, c\}^* \mid |w|_b = |w|_c \}.
\]

The positive equivalence classes of \( \sim_L \) are \([\lambda], [a], \) and \([bc]\). So, the language \( L \) is in \( \text{ROWJ} \), by Corollary 4. Let the \( \lambda \)-free homomorphism \( h : \{a, b, c\}^* \to \{a, b\}^* \) be defined by \( h(a) = a, h(b) = b, \) and \( h(c) = a \). Then, we get

\[
h(L) = \{ w \in \{a, b\}^* \mid |w|_b = 0 \lor |w|_b = |w|_a \},
\]

which was shown to be not in \( \text{ROWJ} \) in Lemma 5.

We also consider the operation of inverse homomorphism:

**Theorem 28.** The family \( \text{ROWJ} \) is not closed under inverse homomorphism.
Proof. Let $A$ be the ROWJFA $A = \{(q_0, q_1, q_2), \{a, b, c\}, R, q_0, \{q_0, q_2\}\}$, where $R$ consists of the rules $q_0c \rightarrow q_0$, $q_0b \rightarrow q_1$, $q_1a \rightarrow q_2$, and $q_2b \rightarrow q_1$. The ROWJFA $A$ is depicted in Figure 3.

![Figure 3. The ROWJFA $A$ satisfying $L(A) \cap \{ac,b\}^* = \{(ac)^nb^n \mid n \geq 0\}$.](image)

Let $h : \{a, b\}^* \rightarrow \{a, b, c\}^*$ be the homomorphism, given by $h(a) = ac$ and $h(b) = b$. It is not hard to see that $h(\{a, b\}^*) = \{ac, b\}^*$.

Let now $\lambda \neq w \in L(A) \cap \{ac, b\}^*$, which implies $|w|_b > 0$. When $A$ reads $w$, it reaches the first occurrence of the symbol $b$ in state $q_0$. After reading this $b$, the automaton is in state $q_1$. Now, no more $c$ can be read. So, we get $w \in (ac)^+b^\dagger$. Whenever $A$ is in state $q_2$, it has read the same number of $a$’s and $b$’s. This gives us $w \in \{(ac)^nb^n \mid n > 0\}$. That shows the inclusion of $L(A) \cap \{ac, b\}^*$ within $\{(ac)^nb^n \mid n \geq 0\}$.

On the other hand, for $n > 0$, we have

$$q_0(ac)^nb^n \cup^n q_0b^na^n \cup^2 q_2a^{n-1}b^{n-1} \cup^2 q_2a^{n-2}b^{n-2} \cup^2 \cdots \cup^2 q_2ab \cup^2 q_2.$$

This implies $L(A) \cap \{ac, b\}^* = \{(ac)^nb^n \mid n \geq 0\}$. We get

$$h^{-1}(L(A)) = h^{-1}(L(A) \cap h(\{a, b\}^*))$$

$$= h^{-1}(L(A) \cap \{ac, b\}^*)$$

$$= h^{-1}(\{(ac)^nb^n \mid n \geq 0\}) = \{a^nb^n \mid n \geq 0\}.$$

In [3] it was shown that this language is not in $\text{ROWJ}$. \hfill \Box

Finally, we take a look at the permutation closure of $\text{ROWJ}$.

**Theorem 29.** The family $\text{ROWJ}$ is not closed under permutation closure.

**Proof.** By Theorem 11, there is a language $L$, that is in $\text{JFA}$, but not in $\text{ROWJ}$. There exists a DFA $A$ with $L_J(A) = L$. Because of (1), we have $\text{perm}(L_R(A)) = L_J(A) = L$. \hfill \Box

Next, we consider the language family $\text{pROWJ}$ in more detail. One can easily find witness languages to see that $\text{pROWJ}$ is not closed under union with regular languages, intersection with regular languages, concatenation, concatenation with regular languages (from both sides), Kleene star, Kleene plus, substitution, homomorphism, and $\lambda$-free homomorphism. For all these operations, the witness languages can be chosen in a way such that the resulting language is not even permutation closed. On the other hand, it is not hard to see that the family of permutation closed languages is closed under union, intersection, complement, and inverse homomorphism. We investigate how the language family $\text{pROWJ}$ behaves under the latter four operations. From the proofs of the Theorems 21 and 22 we get:

**Theorem 30.** The family $\text{pROWJ}$ is not closed under union and under complement.

The next theorem shows that $\text{pROWJ}$ is closed under intersection.

**Theorem 31.** Let $L_1 \in \text{pROWJ}_m$ and $L_2 \in \text{pROWJ}_n$, for some $n, m \geq 0$. Then, the language $L_1 \cap L_2 \in \text{pROWJ}_{mn}$.
Proof. Let $\Sigma$ be an alphabet such that $L_1, L_2 \subseteq \Sigma^*$. The set $\Sigma^*$ is enumerable, so, there is a total order on $\Sigma^*$ such that each non-empty subset of $\Sigma^*$ has exactly one minimal element. Set

$$X = \bigcup_{(S,T) \in (L_1/\sim L_1) \times (L_2/\sim L_2)} \{\min(S \cap T)\}.$$ 

Because of Theorem 3 we have $|L_1/\sim L_1| \leq m$ and $|L_2/\sim L_2| \leq n$. That gives us $|X| \leq mn$. Now, let $w \in L_1 \cap L_2$. There exists exactly one

$$(S,T) \in (L_1/\sim L_1) \times (L_2/\sim L_2)$$

such that $w \in S \cap T$. Let $v = \min(S \cap T)$ and $u$ be an arbitrary word in $\Sigma^*$. For $i \in \{1,2\}$, we have $wu \in L_i$ if and only if $vu \in L_i$, because of $w \sim L_i v$. This implies that $wu \in L_1 \cap L_2$ if and only if $vu \in L_1 \cap L_2$. We get $w \sim L_1 \cap L_2$ and $\sim L_1 \cap L_2$. Thus we immediately get:

$$|L_1/\sim L_1| \leq m \text{ and } |L_2/\sim L_2| \leq n.$$ 

As an immediate consequence we get:

**Corollary 32.** The family $p\text{ROWJ}$ is closed under intersection.

Our next result implies that $p\text{ROWJ}$ is closed under inverse homomorphism.

**Theorem 33.** Let $\Gamma$ and $\Sigma$ be alphabets and $h : \Gamma^* \rightarrow \Sigma^*$ be a homomorphism. Furthermore let $L \subseteq \Sigma^*$ be in $p\text{ROWJ}_n$, for some $n \geq 0$. Then, the language $h^{-1}(L)$ is also in $p\text{ROWJ}_n$.

Proof. Theorem 3 gives us $|L/\sim L| \leq n$. From $L = \bigcup_{S \in L/\sim L} S$, we get

$$h^{-1}(L) = \bigcup_{S \in L/\sim L} h^{-1}(S).$$

Consider now an element $S \in L/\sim L$, two words $v, w \in h^{-1}(S)$, and an arbitrary $u \in \Gamma^*$. Because of $h(v), h(w) \in S$, we have $h(v) \sim_L h(w)$. It follows that

$$vu \in h^{-1}(L) \iff h(v)h(u) \in L \iff h(w)h(u) \in L \iff vu \in h^{-1}(L).$$

We have shown $v \sim_{h^{-1}(L)} w$. So, we get $|h^{-1}(L)/\sim_{h^{-1}(L)}| \leq |L/\sim L| \leq n$, which implies that $h^{-1}(L)$ is in $p\text{ROWJ}_n$, by Theorem 3.

Thus we immediately get:

**Corollary 34.** The family $p\text{ROWJ}$ is closed under inverse homomorphism.

6 More on Languages Accepted by ROWJFAs

In Corollary 4 a characterization of the permutation closed languages that are in $\text{ROWJ}$ was given. In this section, we characterize languages in $\text{ROWJ}$ for some cases where the considered language does not need to be permutation closed.

**Theorem 35.** For an alphabet $\Sigma$, let $w \in \Sigma^*$ and $L \subseteq \Sigma^*$. Then, the language $wL$ is in $\text{ROWJ}$ if and only if $L$ is in $\text{ROWJ}$. 

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Lemma 37. Let \( wL \) be a language with only a finite number of positive equivalence classes. Then, the language \( wL \) shown that
\[\text{for an alphabet } \Sigma, \text{ we need the following lemma. It treats the case of an ROWJFA that is only allowed to jump over one of the input symbols.} \]

Proof. If \( L \) is in \( \text{ROWJ} \), then \( wL \) is also in \( \text{ROWJ} \), because the language family \( \text{ROWJ} \) is closed under concatenation with prefix-free languages from the left. Now assume that \( wL \) is in \( \text{ROWJ} \) and \( L \neq \emptyset \). We may also assume that \( |w| = 1 \). The general case follows from this special case via a trivial induction over the length of \( w \). Thus, let \( w = a \) for an \( a \in \Sigma \) and let \( A = (Q, \Sigma, R, s, F) \) be a DFA with \( L_R(A) = aL \). In the following, we will show via a contradiction that the value \( R(s, a) \) is defined. Assume that \( R(s, a) \) is undefined and let \( v \) be an arbitrary word out of \( L \). Because \( av \in L_R(A) \), there is a symbol \( b \in \Sigma_a \), two words \( x \in (\Sigma \setminus \Sigma_a)^* \) and \( y \in \Sigma^* \), and a state \( p \in F \) such that \( v = xyb \) and \( saxb \in R(s, b)yax \in \Sigma^+ \). This gives us \( sbxay \vdash R(s, b)yax \in \Sigma^+ \), which implies \( byax \in L_R(A) = aL \). However, this is a contradiction, because \( b \neq a \). So, the value \( R(s, a) \) is defined.

Consider the DFA \( B = (Q, \Sigma, R, R(s, a), F) \). For a word \( z \in \Sigma^* \), we have \( z \in L_R(B) \) if and only if \( az \in L_R(A) = aL \), because of \( saz \vdash R(s, a)z \). That gives us \( L_R(B) = L \) and we have shown that \( L \) is in \( \text{ROWJ} \).

From the previous theorem and Corollary 4 we get a generalization of the latter corollary.

Corollary 36. For an alphabet \( \Sigma \), let \( w \in \Sigma^* \) and let \( L \subseteq \Sigma^* \) be a permutation closed language. Then, the language \( wL \) is in \( \text{ROWJ} \) if and only if the Myhill-Nerode equivalence relation \( \sim_L \) has only a finite number of positive equivalence classes.

Next, we will give a characterization for the concatenation \( Lw \) of a language \( L \) and a word \( w \). To do so, we need the following lemma. It treats the case of an \( \text{ROWJFA} \) that is only allowed to jump over one of the input symbols.

Lemma 37. Let \( A = (Q, \Sigma, R, s, F) \) be a DFA with a symbol \( a \in \Sigma \) such that the value \( R(q, b) \) is defined for all \( (q, b) \in Q \times (\Sigma \setminus \{a\}) \). Then, the language \( L_R(A) \) is regular.

Proof. Consider the DFA
\[B = (Q \times (Q \cup \{d\})^Q, \Sigma, S, (s, \text{id}_{Q \cup \{d\}}, G), G = \{(q, f) \in Q \times (Q \cup \{d\})^Q \mid f(q) \in F \}. \]
The total map \( S \) is defined as follows:
\[S((q, f), b) = \begin{cases} (R(q, b), f) & \text{if } (q, f) \in Q \times (Q \cup \{d\})^Q \text{ and } b \in \Sigma_{R,q} \\ (q, g \circ f) & \text{if } (q, f) \in Q \times (Q \cup \{d\})^Q \text{ and } b \notin \Sigma_{R,q}. \end{cases} \]
The map \( g : (Q \cup \{d\}) \to (Q \cup \{d\}) \) is defined in the following way: let
\[g(p) = \begin{cases} R(p, a) & \text{if } p \in Q \text{ and } a \in \Sigma_{R,p} \\ d & \text{if } p \in Q \text{ and } a \notin \Sigma_{R,p}. \end{cases} \]
This completes the description of \( B \). We will show that \( L_D(B) = L_R(A) \).

Let \( w \in \Sigma^* \). We decompose the word \( w \) into factors that are consumed by \( A \) and factors that are jumped over by the automaton in question: there exists a number \( m > 0 \),
words $w_1, w_2, \ldots, w_m \in \Sigma^*$, symbols $b_2, b_3, \ldots, b_m \in \Sigma \setminus \{a\}$, numbers $n_1, \ldots, n_m \in \mathbb{N}$ such that $n_1n_2 \cdots n_{m-1} > 0$, and states $p_1, p_2, \ldots, p_m \in Q$ and $q_2, q_3, \ldots, q_m \in Q$ with

$$sw = sw_1a^{n_1} \prod_{i=2}^{m} b_iw_ia^{n_i} \vdash A^{|w_1|} p_1a^{n_1} \prod_{i=2}^{m} b_iw_ia^{n_i}$$

$$\bigcirc_A q_2aw_2a^{n_2} \left( \prod_{i=3}^{m} b_iw_ia^{n_i} \right) a^{n_1}$$

$$\vdash |w_2| p_2a^{n_2} \left( \prod_{i=3}^{m} b_iw_ia^{n_i} \right) a^{n_1}$$

$$\bigcirc_A q_3aw_3a^{n_3} \left( \prod_{i=4}^{m} b_iw_ia^{n_i} \right) a^{n_1+n_2}$$

$$\vdots$$

$$\vdash |w_{m-1}| p_{m-1}a^{n_{m-1}}b_mw_ma^{n_m+n_{m-2}}$$

$$\bigcirc_A q_ma^{n_m} \prod_{i=1}^{m} w_m^{n_i}$$

We have $w \in L_R(A)$ if and only if $g_{\sum_{i=1}^{m} n_i}(p_m) \in F$. On the other hand, we get the following computation

$$(s, \text{id}_{Q \cup \{d\}})w = (s, \text{id}_{Q \cup \{d\}})w_1a^{n_1} \prod_{i=2}^{m} b_iw_ia^{n_i}$$

$$\vdash |w_1| (p_1, \text{id}_{Q \cup \{d\}})a^{n_1} \prod_{i=2}^{m} b_iw_ia^{n_i}$$

$$\vdash_{n_1+1} B (q_2, g^{n_1}|Q)w_2a^{n_2} \left( \prod_{i=3}^{m} b_iw_ia^{n_i} \right)$$

$$\vdash_{n_2+1} B (p_2, g^{n_1}|Q)a^{n_2} \left( \prod_{i=3}^{m} b_iw_ia^{n_i} \right)$$

$$\vdash_{n_3+1} B (q_3, g^{n_1+n_2}|Q)w_3a^{n_3} \left( \prod_{i=4}^{m} b_iw_ia^{n_i} \right)$$

$$\vdots$$

$$\vdash_{n_{m-1}} B (p_{m-1}, g^{\sum_{i=1}^{m-1} n_i}|Q)a^{n_{m-1}}b_mw_ma^{n_m}$$

$$\vdash_{n_{m-1}+1} B (q_m, g^{\sum_{i=1}^{m-1} n_i}|Q)w_ma^{n_m}$$

$$\vdash_{n_m} B (p_m, g^{\sum_{i=1}^{m-1} n_i}|Q)a^{n_m}.$$

Set $k = \max \{ r \in \{0, 1, \ldots, n_m\} \mid g^r(p_m) \in Q \}$. That gives

$$(p_m, g^{\sum_{i=1}^{m-1} n_i}|Q)a^{n_m} \vdash_B (g^k(p_m), g^{\sum_{i=1}^{m-1} n_i}|Q)a^{n_m-k}$$

$$\vdash_B (g^{k-1}(p_m), g^{\sum_{i=1}^{m-1} n_i-k}|Q).$$

Thus, we have $w \in L_D(B)$ if and only if

$$g_{\sum_{i=1}^{m} n_i}(p_m) = g_{\sum_{i=1}^{m} n_i-k}(g^k(p_m)) \in F,$$
which holds if and only if \( w \in L_R(A) \). That shows \( L_D(B) = L_R(A) \) and that \( L_R(A) \) is a regular language. This proves the lemma.

Our characterization for languages of the form \( Lw \) generalizes a result from [3], which says that the language \( \{ va | v \in \{ a, b \}^*, |v|_a = |v|_b \} \) is not in \( \text{ROWJ} \):

**Theorem 38.** For an alphabet \( \Sigma \), let \( w \in \Sigma^* \) be a non-empty word and \( L \subseteq \Sigma^* \). Then, the language \( Lw \) is in \( \text{ROWJ} \) if and only if \( L \) is regular.

**Proof.** If \( L \) is regular, then \( Lw \) is also regular, which means that \( Lw \) is in \( \text{ROWJ} \). Assume now, that \( Lw \) is in \( \text{ROWJ} \). As in the proof of Theorem 35, we can assume that \( w = a \) for an \( a \in \Sigma \).

Let \( A = (Q, \Sigma, R, s, F) \) be a DFA with \( L_R(A) = L_a \). Consider the DFA \( B = (Q \cup \{d\}, \Sigma, S, s, F) \) and let \( d \) be a new symbol with \( d \notin Q \cup \Sigma \). The map \( S \) is defined as follows: for \( (q, b) \in Q \times \Sigma \), we set \( S(q, b) = R(q, b) \), if \( R(q, b) \) is defined. If \( R(q, b) \) is undefined and \( b \neq a \), we define \( S(q, b) = d \).

For all \( q \in Q \), the value \( S(q, a) \) is undefined, if \( R(q, a) \) is undefined. Finally, for all \( b \in \Sigma \), it holds \( S(d, b) = d \). By Lemma 37, the language \( L_R(B) \) is regular. We will show that

\[
L_R(A) = L_R(B).
\]

Then, the regularity of \( L_a = L_R(A) \) implies the regularity of \( L \), because regular languages are closed under the operation of quotient with a regular language.

First, let \( v \in L_R(B) \) and \( f \in F \) with \( sv \cup_B f \). For a state \( q \in Q \), a symbol \( b \in \Sigma_{S,q} \), and words \( x \in (\Sigma \setminus \Sigma_{S,q})^* \) and \( y \in \Sigma^* \) with \( sv \cup_B f \), we have \( x \in (\Sigma \setminus \Sigma_{R,a})^* \) and

\[
qxby \cup_B f = S(q, b)y \cup_B f.
\]

This implies \( S(q, b) \neq d \), which tells us \( b \in \Sigma_{R,a} \) and \( R(q, b) = S(q, b) \). We get \( qxby \cup_A f \). By induction, we see that \( sv \cup_A f \). Therefore, we have \( v \in L_R(A) \).

Now, let \( v \in L_R(A) \) and \( f \in F \) with \( sv \cup_A f \). Assume that \( v \notin L_R(B) \). Then, there exists a symbol out of \( \Sigma \setminus \{a\} \) that is jumped over during the processing of \( A \), when the starting configuration is \( sv \). The part of \( v \) that is visited by \( A \) before it jumps over the first symbol out of \( \Sigma \setminus \{a\} \) will be decomposed into factors that are consumed by \( A \) and factors that are jumped over by the device under consideration: there is a natural number \( m > 0 \), words \( w_1, w_2, \ldots, w_{m+2} \in \Sigma^* \), symbols \( b_2, b_3, \ldots, b_m+1 \in \Sigma \setminus \{a\} \) and \( c \in \Sigma \), numbers \( n_1, \ldots, n_m \in \mathbb{N} \) with \( n_1n_2 \cdots n_{m-1} > 0 \), and states \( p_1, p_2, \ldots, p_m, q_2, q_3, \ldots, q_{m+1} \in Q \) with symbols \( b_{i+1} \in \Sigma_{p_i} \), for every \( i \) satisfying \( i \in \{1, 2, \ldots, m-1\} \), such that word \( a^{n_m}b_{m+1}w_{m+1} \in (\Sigma \setminus \Sigma_{R,p_m})^* \), \( c \in \Sigma_{R,p_m} \), and

\[
sw = sw_1a^{n_1} \left( \prod_{i=2}^{m} b_iw_i a^{n_i} \right) b_{m+1}w_{m+1}c w_{m+2}
\]

\[
\prod_{i=1}^{m} b_iw_i a^{n_i} b_{m+1}w_{m+1}c w_{m+2}
\]

\[
\prod_{i=3}^{m} b_iw_i a^{n_i} b_{m+1}w_{m+1}c w_{m+2} a^{n_1}
\]

\[
\prod_{i=2}^{m} b_iw_i a^{n_i} b_{m+1}w_{m+1}c w_{m+2} a^{n_1}
\]

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continued by
\[
q_2 w_2 a^{n_2} \left( \prod_{i=3}^{m} b_i w_i a^{n_i} \right) b_{m+1} w_{m+1} c w_{m+2} a^{n_1}
\]
\[
\vdash_A |w_2| p_2 a^{n_2} \left( \prod_{i=3}^{m} b_i w_i a^{n_i} \right) b_{m+1} w_{m+1} c w_{m+2} a^{n_1}
\]
\[
\cup_A q_3 w_3 a^{n_3} \left( \prod_{i=4}^{m} b_i w_i a^{n_i} \right) b_{m+1} w_{m+1} c w_{m+2} a^{n_1+n_2}
\]
\[
\vdots
\]
\[
\vdash_A |w_i| p_m a^{n_m} b_{m+1} w_{m+1} c w_{m+2} a^{\sum_{i=1}^{m-1} n_i}
\]
\[
\cup_A q_m w_m a^{n_m} b_{m+1} w_{m+1} c w_{m+2} a^{\sum_{i=1}^{m-1} n_i}
\]
\[
\vdash_A |w| p_m a^{n_m} b_{m+1} w_{m+1} c w_{m+2} a^{\sum_{i=1}^{m} n_i}
\]
\[
\cup_A q_m+1 w_{m+2} a^{\sum_{i=1}^{m} n_i} b_{m+1} w_{m+1} c_A^* f.
\]

We get
\[
sw_1 \left( \prod_{i=2}^{m} b_i w_i \right) w_{m+1} c w_{m+2} a^{\sum_{i=1}^{m} n_i} b_{m+1}
\]
\[
\vdash_A \prod_{i=2}^{m} b_i w_i | p_m w_{m+1} c w_{m+2} a^{\sum_{i=1}^{m} n_i} b_{m+1}
\]
\[
\cup_A q_{m+1} w_{m+2} a^{\sum_{i=1}^{m} n_i} b_{m+1} w_{m+1} c_A^* f.
\]

This implies
\[
w_1 \left( \prod_{i=2}^{m} b_i w_i \right) w_{m+1} c w_{m+2} a^{\sum_{i=1}^{m} n_i} b_{m+1} \in L_R(A) = L a,
\]
a contradiction. This shows \( v \in L_R(B) \), which proves the theorem. \( \square \)

Now, we consider the case of two languages over disjoint alphabets.

**Theorem 39.** For disjoint alphabets \( \Sigma_1 \) and \( \Sigma_2 \), let languages \( L_1 \subseteq \Sigma_1^* \) and \( L_2 \subseteq \Sigma_2^* \) satisfy \( L_1 \neq \emptyset \neq L_2 \neq \{ \lambda \} \) such that \( L_1 L_2 \) is in \textsc{rowj}. Then, the language \( L_1 \) is regular and \( L_2 \) is in \textsc{rowj}.

**Proof.** The proof is similar as those of Theorem 35. Let \( A = (Q, \Sigma, \Sigma_1 \cup \Sigma_2, R, s, F) \) be a DFA with \( L_R(A) = L_1 L_2 \). For an \( m \geq 0 \) and \( a_1, a_2, \ldots, a_m \in \Sigma_1 \), let \( w = a_1 a_2 \ldots a_m \in \Sigma_1 \). We will show by induction that for each \( 0 \leq n \leq m \), there is a state \( q_n \in Q \) with \( sw \vdash^n q_n a_{n+1} a_{n+2} \ldots a_m \). For \( n = 0 \), we just set \( q_0 = s \). Assume that, for a fixed \( k \) with \( 0 \leq k < m \), we already know that there is a state \( q_k \in Q \) with
\[
sw \vdash^k q_k a_{k+1} a_{k+2} \ldots a_m.
\]
If the value \( R(q_k, a_{k+1}) \) is defined, then we have
\[
sw \vdash^{k+1} R(q_k, a_{k+1}) a_{k+2} a_{k+3} \ldots a_m.
\]
Therefore, now let \( R(q_k, a_{k+1}) \) be undefined and let \( v \) be an arbitrary non-empty word out of \( L_2 \). We get
\[
sw v \vdash^k q_k a_{k+1} a_{k+2} \ldots a_m v.
\]
Because of \( vw \in L_R(A) \), there exist a symbol \( b \in \Sigma_{q_k} \), words \( x \in (\Sigma \setminus \Sigma_{q_k})^* \) and \( y \in \Sigma^* \), and a state \( p \in F \) such that \( a_{k+2}a_{k+3}\ldots a_{m}v = xby \) and

\[
q_{k}a_{k+1}xby \circ R(q_k, b)ya_{k+1}x \circ^* p.
\]

This implies

\[
sa_1a_2\ldots a_kxbya_{k+1} \vdash^k q_{k}xbya_{k+1} \circ R(q_k, b)ya_{k+1}x \circ^* p,
\]

which gives us that \( a_1a_2\ldots a_kxbya_{k+1} \in L_R(A) \). However, this word is equal to

\[
a_1a_2\ldots a_ka_{k+2}a_{k+3}\ldots a_{m}va_{k+1}
\]

and we have

\[
a_1a_2\ldots a_ka_{k+2}a_{k+3}\ldots a_{m}va_{k+1} \in \Sigma^* \setminus \Sigma_{1}^* \Sigma_{2}^* \subseteq \Sigma^* \setminus (\Sigma_{1}^* \Sigma_{2}^*) \subseteq \Sigma^* \setminus (L_1L_2),
\]

which is a contradiction. So, the value \( R(q_k, a_{k+1}) \) has to be defined and we have shown by induction that for each \( 0 \leq n \leq m \), there is a state \( q_n \in Q \) with \( sw \vdash^n q_n a_{n+1}a_{n+2}\ldots a_{m} \). We set \( q_w = q_m \) and get \( sw \vdash^{|w|} q_w \).

For every \( w \in L_1 \), we consider the DFA \( B_w = (Q, \Sigma_2, R|_{Q \times \Sigma_2}, q_w, F) \). For every \( v \in \Sigma_2^* \), we have

\[
v \in L_R(B_w) \iff (\exists f \in F : q_w v \circ^*_w f)
\]

\[
\iff (\exists f \in F : swv \circ^*_A f)
\]

\[
\iff wv \in L_R(A) = L_1L_2
\]

\[
\iff v \in L_2.
\]

This shows \( L_R(B_w) = L_2 \), so \( L_2 \) is in \textbf{ROWJ}.

For every \( v \in L_2 \), we define the set \( Q_v = \{ q \in Q \mid \exists f \in F : qv \circ^*_A f \} \) and the deterministic finite state device \( C_v = (Q, \Sigma_1, R|_{Q \times \Sigma_1}, s, Q_v) \). For every \( w \in \Sigma_1^* \), we have

\[
w \in L_D(C_v) \iff (\exists q \in Q_v : sw \vdash^*_v q)
\]

\[
\iff (\exists q \in Q, f \in F : swv \vdash^*_A qv \circ^*_A f)
\]

\[
\iff wv \in L_R(A) = L_1L_2
\]

\[
\iff w \in L_1.
\]

Therefore, we conclude \( L_D(C_v) = L_1 \), so \( L_1 \) is regular, which proves the theorem. \( \Box \)

Adding prefix-freeness for \( L_1 \), we get an equivalence, by Theorem 39 and the closure of \textbf{ROWJ} under left-concatenation with prefix-free regular sets.

**Corollary 40.** For disjoint alphabets \( \Sigma_1 \) and \( \Sigma_2 \), let \( L_1 \subseteq \Sigma_1^* \) be a prefix-free set and \( L_2 \subseteq \Sigma_2^* \) be an arbitrary language with \( L_1 \neq \emptyset \neq L_2 \neq \{\lambda\} \). Then, the language \( L_1L_2 \) is in \textbf{ROWJ} if and only if \( L_1 \) is regular and \( L_2 \) is in \textbf{ROWJ}.

The previous corollary directly implies the following characterization that is another generalization of Corollary 4.

**Corollary 41.** For disjoint alphabets \( \Sigma_1 \) and \( \Sigma_2 \), let \( L_1 \subseteq \Sigma_1^* \) be a prefix-free set and \( L_2 \subseteq \Sigma_2^* \) be a permutation closed language with \( L_1 \neq \emptyset \neq L_2 \neq \{\lambda\} \). Then, the language \( L_1L_2 \) is in \textbf{ROWJ} if and only if \( L_1 \) is regular and the Myhill-Nerode equivalence relation \( \sim_{L_2} \) has only a finite number of positive equivalence classes.
If a non-empty language and a non-empty permutation closed language over disjoint alphabets are separated by a symbol, we get the following result:

**Corollary 42.** For disjoint alphabets $\Sigma_1$ and $\Sigma_2$, let $L_1 \subseteq \Sigma_1^*$ be non empty and $L_2 \subseteq \Sigma_2^*$ be a non-empty permutation closed language. Furthermore, let $a \in \Sigma_2$. Then, the language $L_1alL_2$ is in $\text{ROWJ}$ if and only if $L_1$ is regular and the Myhill-Nerode equivalence relation $\sim_{L_2}$ has only a finite number of positive equivalence classes.

**Proof.** If $L_1$ is regular and the Myhill-Nerode equivalence relation $\sim_{L_2}$ has only a finite number of positive equivalence classes, Corollary 26 tells us that $L_1alL_2$ is in $\text{ROWJ}$. Now, assume that $L_1alL_2$ is in $\text{ROWJ}$. From Theorem 39 we get that $L_1$ is regular and $aL_2$ is in $\text{ROWJ}$. Corollary 36 gives us that $\sim_{L_2}$ has only a finite number of positive equivalence classes.

For an alphabet $\Sigma = \{a_1, a_2, \ldots, a_n\}$, the family of subsets of $a_1^*a_2^* \ldots a_n^*$ is kind of a counterpart of the family of permutation closed languages over $\Sigma$. In a permutation closed language $L$, for each word $w \in L$, all permutations of $w$ are also in $L$. In a language $M \subseteq a_1^*a_2^* \ldots a_n^*$, for each word $w \in M$, no other permutation of $w$ is in $M$. We can characterize the subsets of $a_1^*a_2^* \ldots a_n^*$ that are in $\text{ROWJ}$. The following lemma helps us to do so.

**Lemma 43.** Let $A$ be a DFA with input alphabet $\{a_1, a_2, \ldots, a_n\}$ accepting a letter bounded language, i.e., $L_R(A) \subseteq a_1^*a_2^* \ldots a_n^*$. Then, $L_R(A) = L_D(A)$.

**Proof.** Because of (1) we have $L_D(A) \subseteq L(R(A))$. Now assume that $w \in L(R(A))$. Again, because of this inclusion chain, there is a permutation $v$ of the word $w$ with $v \in L_D(A) \subseteq L(R(A))$. Since $L_R(A) \subseteq a_1^*a_2^* \ldots a_n^*$, we conclude that $w = v \in L_D(A)$. Thus, $L_R(A) = L_D(A)$.

In [3] it was shown that the language $\{a^n b^n | n \geq 0\}$ is not in $\text{ROWJ}$. Our characterization generalizes this result:

**Theorem 44.** Let $\{a_1, a_2, \ldots, a_n\}$ be an alphabet and $L \subseteq a_1^*a_2^* \ldots a_n^*$. Then, the language $L$ is in $\text{ROWJ}$ if and only if $L$ is regular.

**Proof.** If $L$ is regular, then $L$ is also in $\text{ROWJ}$, because of $\text{REG} \subseteq \text{ROWJ}$. If $L$ is in $\text{ROWJ}$, then there exists a DFA $A$ with $L = L_R(A)$. Because of Lemma 43, we get $L = L_R(A) = L_D(A)$, which is a regular language. This proves the stated claim.

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