I F I G
RESEARCH
REPORT

INSTITUT FÜR INFORMATIK

ECONOMY OF DESCRIPTION FOR BASIC CONSTRUCTIONS ON RATIONAL TRANSDUCTIONS

Henning Bordihn Markus Holzer Martin Kutrib

IFIG RESEARCH REPORT 0204
JULY 2002

Institut für Informatik
JLU Gießen
Arndtstraße 2
D-35392 Giessen, Germany
Tel: +49-641-99-32141
Fax: +49-641-99-32149
mail@informatik.uni-giessen.de
www.informatik.uni-giessen.de

JUSTUS-LIEBIG-
UNIVERSITÄT
GIESSEN

IFIG Research Report
IFIG Research Report 0204, July 2002

Economy of Description for Basic Constructions on Rational Transductions

Henning Bordihn[1]

Institut für Informatik, Universität Potsdam
August-Bebel-Straße 89, D-14482 Potsdam, Germany

Markus Holzer[2]

Institut für Informatik, Technische Universität München
Boltzmannstraße 3, D-85748 Garching bei München, Germany

Martin Kutrib[3]

Institut für Informatik, Universität Giessen
Arndtstraße 2, D-35392 Giessen, Germany

Abstract. The state complexities of basic constructions on rational transductions are investigated. Given rational transductions, described by rational transducers, and an operation thereon we consider the number of states that is sufficient and necessary in the worst case to describe the resulting transduction. In particular, tight bounds are shown for inversion, union, weak intersection, concatenation, reversal, homomorphism, and composition, i.e., for some of those operations under which rational transductions are closed.

CR Subject Classification (1998): F.1, F.4.3

---

[1]E-mail: henning@cs.uni-potsdam.de
[2]E-mail: holzer@informatik.tu-muenchen.de
[3]E-mail: kutrib@informatik.uni-giessen.de

1 Introduction

Due to several applications and implementations of transducers in theoretical and practical areas of computer science their descriptional complexity is a natural question of crucial importance. The applications are widely spread. For example, to compiler constructions [1], language and speech processing [8], and even to the design of controllability systems in aircraft design [10]. Much of the underlying theory has originated from linguistics. In natural language and speech processing transducers with more than one hundred million states may be used.

Whereas the state complexity of finite automata has been investigated by several authors in detail [15] only little is known for transducers. From an algorithmic point of view minimization of sequential transducers is considered in [9]. Papers dealing explicitly with state complexity are [12, 13] where certain types of transducers are transformed to others. Here we are investigating the state complexity of basic constructions on rational transductions. This is, given rational transductions (or relations) described by rational transducers and an operation thereon, we consider the number of states that is sufficient and necessary in the worst case to describe the resulting transduction. Certainly, this requires the closure of transductions under the operations. Rational transducers are nondeterministic devices whose transductions are closed under inversion [14], union, intersection on the first or second component, concatenation, reversal, homomorphism, and composition [5]. The deterministic variant is solely closed under composition. From these facts it follows that there exist essentially nondeterministic transductions.

Comparing rational transductions with nondeterministic finite automata [6] one observes that the transducers accept and transform simultaneously. The undecidability of the equivalence problem for rational transducers has been shown in [7] while the problem becomes decidable again for single-valued rational transducers [4, 11]. Therefore the constructions are expected to be more expensive. Nevertheless, it turns out that for some operations defined on both devices the bounds are similar.

The technical depth of our results varies from immediate to more subtle extensions to previous work. Indeed the technique to prove minimality for deterministic finite automata is not directly applicable to the case of rational transducers. Therefore, we mostly have to use counting arguments to prove our results on rational transducer minimality with respect to the number of states.

The paper is organized as follows. In the next section we define the basic notions. In Section 3 we prove tight bounds for the mentioned operations. For example, the composition of an n-state and an m-state transducer needs $m \cdot n$ states in the worst case. From the state complexity point of view it has a favourable effect to handle the $(m+n)$-state decomposition of the $(m \cdot n)$-state transducer instead of the transducer itself. At the end of the section a table is given that summarizes the presented results.

2 Preliminaries

We denote the set of positive integers $\{1, 2, \dots\}$ by $\mathbb{N}$, the set $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$, and the powerset of a set S by 2^S. The empty word is denoted by λ and the reversal of a word w by w^R. For the length of w we write $|w|$. The number of occurrences of a symbol a in the word w is denoted by $\#_a(w)$. We use $\subseteq$ for inclusions and $\subset$ if the inclusions are strict.

A rational transducer is a finite-state transformation device possessing an input tape and an output tape. At each time step the transducer reads a symbol or the empty word from the input tape in some internal state, goes nondeterministically into another state, and writes a symbol or the empty word to the output tape. More formally:

Definition 1 *A* rational transducer (RTD) *is a system* $\langle S, A, B, \delta, s_0, F\rangle$ *where*

1. S *is the finite nonempty set of* internal states,
2. A *is the finite set of* input symbols,
3. B *is the finite set of* output symbols,
4. $s_0 \in S$ *is the* initial state,
5. $F \subseteq S$ *is the set of* accepting states, *and*
6. δ *is the* partial transition function *mapping from* $S \times (A \cup \{\lambda\})$ *into the subsets of* $S \times (B \cup \{\lambda\})$.

The set of rejecting states is implicitly given by the partitioning, i.e., $S \setminus F$.

In some sense the transition function is complete. Without loss of generality we may require δ to be a total function, since whenever the operation of an RTD is supposed not to be defined, then δ can map to the empty set which, trivially, belongs to $2^{S\times(B\cup\{\lambda\})}$. Thus, in some sense the RTD need not be complete. The mode of operation can be generalized such that an RTD reads a whole input word and emits a whole output word. This model yields the same rational transductions. Since for state complexity issues there is a difference (cf. Section 3.6) here we use the single symbol mode which is called standard form in [14].

A *configuration* of an RTD $\mathcal{A}$ is a description of its global state which is a triple (s, u, v) where $s \in S$ is the current state, $u \in A^*$ is the still unread portion of the input, and $v \in B^*$ is the output word produced so far. We write $(s, au, v) \vdash (s', u, vb)$ iff $(s', b) \in \delta(s, a)$. The reflexive and transitive closure of $\vdash$ is denoted by $\vdash^*$, the transitive closure by $\vdash^+$, and thus $(s, u, v) \vdash^* (s', u', v')$ indicates that it is possible for $\mathcal{A}$ to go from the configuration (s, u, v) to the configuration (s', u', v') in a sequence of zero or more moves.

A rational transducer $\mathcal{A}$ *transforms* input words $u \in A^*$ into sets of output words. For a successful transformation $\mathcal{A}$ has to be in an accepting state after having read the whole input, otherwise the output is not recorded:

$$\mathcal{A}(u) = \{v \mid (s_0, u, \lambda) \vdash^* (s_f, \lambda, v), s_f \in F\}$$

The *transduction realized by* $\mathcal{A}$, denoted by $T(\mathcal{A})$, is the set of pairs $(u, v) \in (A^* \times B^*)$ such that $v \in \mathcal{A}(u)$.

If we build the projection on the first components of $T(\mathcal{A})$, denoted by $L(T(\mathcal{A}))$, then the rational transducer degenerates to a nondeterministic finite state acceptor (NFA). The observation that the number of states which is necessary for an NFA to accept $L(T(\mathcal{A}))$ gives a lower bound for the number of states necessary to realize the transduction $T(\mathcal{A})$ bridges both worlds. In the sequel we utilize this bridge in order to simplify proofs and cross it some times.

An RTD is said to be *minimal* if its number of states is minimal with respect to the realized transduction. If not otherwise stated throughout the paper we assume that the RTDs are always minimal. In particular, this implies that there are no unreachable states and that from any state a final state can be reached.

3 Basic Constructions

We start our investigations in the next subsection with the inversion since the operation and its state complexity can serve as helpful tool in the sequel.

3.1 Inversion

Let T be a transduction realized by some RTD. The *inverse transduction* T^{-1} *of* T is defined to be

$$T^{-1} = \{(u, v) \mid (v, u) \in T\}$$

The closure of rational transductions under inversion and the corresponding constructions are mentioned without proof in [14].

Theorem 2 *For any integer $n \in \mathbb{N}$ let $\mathcal{A}$ be an n-state* RTD. *Then n states are sufficient and necessary in the worst case for an* RTD *$\mathcal{C}$ to realize the transduction $T(\mathcal{A})^{-1}$.*

Proof. The construction is done by interchanging the input and output of the transducer $\mathcal{A} = \langle S, A, B, \delta_A, s_0, F\rangle$. To this end define $\mathcal{C} = \langle S, B, A, \delta, s_0, F\rangle$ such that for all $s \in S$ and $b \in B \cup \{\lambda\}$:

$$\delta(s, b) = \{(s', a) \mid (s', b) \in \delta_A(s, a)\}$$

In order to show the correctness of the construction let

$$(s_0, a_1 \cdots a_p, \lambda) \vdash (s_1, u_1, v_1) \vdash \cdots \vdash (s_m, \lambda, b_1 \cdots b_q)$$

with $m \in \mathbb{N}_0$ and $s_m \in F$ be a computation of $\mathcal{A}$. We obtain

$$(a_1 \cdots a_p, b_1 \cdots b_q) \in T(\mathcal{A})$$

Now let $(s_0, b_1 \cdots b_q, \lambda) \vdash (\bar{s}_1, w_1, x_1) \vdash \cdots \vdash (\bar{s}_m, w_m, x_m)$ be a computation of $\mathcal{C}$ under input $b_1 \cdots b_q$. We are going to show that $s_i = \bar{s}_i$, $x_i u_i = a_1 \cdots a_p$ and $v_i w_i = b_1 \cdots b_q$ holds at any time $0 \leq i \leq m$.

In this case we obtain immediately $\bar{s}_m \in F$, $w_m = \lambda$, $x_m = a_1 \cdots a_p$, and thus $(b_1 \cdots b_q, a_1 \cdots a_p) \in T(\mathcal{C})$.

The claim is trivially true for $i = 0$. Concluding inductively, assume it is true for some $0 \leq i < m$.

The $(i + 1)$st transition of $\mathcal{A}$ may be a λ-transition or it may consume the first symbol a_j of u_i. The transition may emit λ or the last symbol b_k of v_{i+1}. So we are concerned with four cases for the $(i + 1)$st configuration of $\mathcal{A}$: $(s_{i+1}, a_j \cdots a_p, b_1 \cdots b_{k-1})$, $(s_{i+1}, a_{j+1} \cdots a_p, b_1 \cdots b_{k-1})$, $(s_{i+1}, a_j \cdots a_p, b_1 \cdots b_k)$ resp. $(s_{i+1}, a_{j+1} \cdots a_p, b_1 \cdots b_k)$. By construction it follows that $(s_{i+1}, \lambda) \in \delta(s_i, \lambda)$, $(s_{i+1}, a_j) \in \delta(s_i, \lambda)$, $(s_{i+1}, \lambda) \in \delta(s_i, b_k)$ resp. $(s_{i+1}, a_j) \in \delta(s_i, b_k)$. Therefore by induction hypothesis we obtain $\bar{s}_{i+1} = s_{i+1}$, and $x_{i+1} = x_i$, $x_{i+1} = x_i a_j$, $x_{i+1} = x_i$ resp. $x_{i+1} = x_i a_j$, and $w_i = w_{i+1}$, $w_i = w_{i+1}$, $w_i = b_k w_{i+1}$ resp. $w_i = b_k w_{i+1}$. Hence, the claim and therefore $T(\mathcal{A}) \subseteq T(\mathcal{C})$ follows.

Since $(s', a) \in \delta(s, b) \iff (s', b) \in \delta_A(s, a)$ the converse $T(\mathcal{C}) \subseteq T(\mathcal{A})$ follows analogously.

Trivially the construction of $\mathcal{C}$ preserves the number n of states of $\mathcal{A}$. On the other hand, n states are necessary if $\mathcal{A}$ is minimal. Otherwise the construction $(T(\mathcal{A})^{-1})^{-1}$ would lead to less than n states. Then $(T(\mathcal{A})^{-1})^{-1} = T(\mathcal{A})$ is a contradiction to the minimality of $\mathcal{A}$.

3.2 Union

Now we turn to the sole Boolean operation under which rational transductions are closed.

Theorem 3 *For any integers $m, n \geq 1$ let $\mathcal{A}$ be an m-state and $\mathcal{B}$ be an n-state* RTD. *Then $m + n + 1$ states are sufficient and necessary in the worst case for an* RTD *$\mathcal{C}$ to realize the transduction $T(\mathcal{A}) \cup T(\mathcal{B})$.*

Proof. In order to construct an appropriate $(m+n+1)$-state RTD we simply use a new initial state and connect it to the initial states of $\mathcal{A}$ and $\mathcal{B}$ via λ-transitions that emit the empty word λ.

Let $\mathcal{A} = \langle S_A, A_A, B_A, \delta_A, s_{0,A}, F_A \rangle$ and $\mathcal{B} = \langle S_B, A_B, B_B, \delta_B, s_{0,B}, F_B \rangle$ with $S_A \cap S_B = \emptyset$, then $\mathcal{C} = \langle S, A_A \cup A_B, B_A \cup B_B, \delta, s_0, F_A \cup F_B \rangle$ is defined as follows:

$$S = S_A \cup S_B \cup \{s_0\} \text{ where } s_0 \notin S_A \cup S_B$$

$$\delta(s, a) = \begin{cases} \{(s_{0,A}, \lambda), (s_{0,B}, \lambda)\} & \text{if } s = s_0 \text{ and } a = \lambda \\ \delta_A(s, a) & \text{if } s \in S_A \\ \delta_B(s, a) & \text{if } s \in S_B \end{cases}$$

for $s \in S$ and $a \in A_A \cup A_B \cup \{\lambda\}$.

During the first transition $\mathcal{C}$ nondeterministically guesses whether the input must be transformed by $\mathcal{A}$ or by $\mathcal{B}$. Subsequently, $\mathcal{A}$ or $\mathcal{B}$ is simulated. Obviously, $T(\mathcal{C}) = T(\mathcal{A}) \cup T(\mathcal{B})$ and $|S| = |S_A| + |S_B| + 1 = m + n + 1$.

In order to show that $m+n+1$ states are necessary in the worst case let $\mathcal{A}$ be an m-state RTD which realizes the transduction $\{(a^{i\cdot m}, \$^i) \mid i \in \mathbb{N}_0\}$ and $\mathcal{B}$ an n-state RTD which realizes $\{(b^{i\cdot n}, \$^i) \mid i \in \mathbb{N}_0\}$. In [6] it has been shown that an NFA needs at least $m+n+1$ states to accept the language $L = \{a^m\}^* \cup \{b^n\}^*$. Since L equals $L(T(\mathcal{A}) \cup T(\mathcal{B}))$ the assertion follows.

3.3 Weak Intersection

The intersection of the rational transductions

$$\{(a^i, b^i c^j) \mid i, j \in \mathbb{N}\} \quad \textit{and} \quad \{(a^i, b^j c^i) \mid i, j \in \mathbb{N}\}$$

is $\{(a^i, b^i c^i) \mid i \in \mathbb{N}\}$ which cannot be realized by an RTD. From the non-closure under intersection and the closure under union the non-closure under complementation, i.e., all pairs over the given alphabets not belonging to the transduction, follows. On the other hand, rational transductions are closed under in some sense weak intersection operations.

Let T_1 and T_2 be two transductions realized by some RTDs. The *intersection for the first components of* T_1 *and* T_2 is defined to be

$$T_1 \cap_1 T_2 = \{(u, v) \mid \exists\, v', v'' : (u, v') \in T_1, (u, v'') \in T_2, v \in \{v', v''\}\}$$

Theorem 4 *For any integers $m, n \in \mathbb{N}$ let $\mathcal{A}$ be an n-state and $\mathcal{B}$ be an m-state* RTD. *Then $2 \cdot m \cdot n + 1$ states are sufficient and necessary in the worst case for an* RTD *$\mathcal{C}$ to realize the transduction $T(\mathcal{A}) \cap_1 T(\mathcal{B})$.*

Proof. Let $\mathcal{A} = \langle S_A, A, B_A, \delta_A, s_{0,A}, F_A \rangle$ and $\mathcal{B} = \langle S_B, A, B_B, \delta_B, s_{0,B}, F_B \rangle$, then $\mathcal{C} = \langle S, A, B_A \cup B_B, \delta, s_0, F \rangle$ is defined as follows. Let S'_A resp. S'_B be copies of S_A resp. S_B, and F'_A resp. F'_B be copies of F_A resp. F_B. Set

$$S = (S_A \times S_B) \cup (S'_A \times S'_B) \cup \{s_0\} \text{ where } s_0 \notin S_A \cup S_B \cup S'_A \cup S'_B,$$
$$F = (F_A \times F_B) \cup (F'_A \times F'_B),$$

$\delta(s_0, \lambda) = \{((s_{0,A}, s_{0,B}), \lambda), ((s'_{0,A}, s'_{0,B}), \lambda)\}$

$\delta((s_1, s_2), a) = \{((t_1, t_2), b) \mid (t_1, b) \in \delta_A(s_1, a) \text{ and } \exists\, b' : \ (t_2, b') \in \delta_B(s_2, a)\}$ for $(s_1, s_2) \in S_A \times S_B$ and $a \in A \cup \{\lambda\}$,

$\delta((s'_1, s'_2), a) = \{((t'_1, t'_2), b') \mid (t_2, b') \in \delta_B(s_2, a) \text{ and } \exists\, b : \ (t_1, b) \in \delta_A(s_1, a)\}$ for $(s'_1, s'_2) \in S'_A \times S'_B$ and $a \in A \cup \{\lambda\}$.

So we did the cross-product construction twice and used a new initial state which is connected to the pairs of old initial states via λ-transitions that emit the empty word.

Now we are going to show that $2mn + 1$ states are necessary in the worst case. Let $\mathcal{A}$ be an m-state RTD which realizes the transduction

$$\{(u, v) \mid u \in \{a, b\}^*, \#_a(u) \equiv 0 \ (\mathrm{mod}\ m), v = b^{\#_a(u)}\}$$

and $\mathcal{B}$ be an n-state RTD which realizes

$$\{(u,v) \mid u \in \{a,b\}^*, \#_b(u) \equiv 0 \pmod{n}, v = a^{\#_b(u)}\}$$

Let $\mathcal{C}$ be a rational transducer for the transduction $T(\mathcal{A}) \cap_1 T(\mathcal{B})$. Since the input $(ab)^i a^{m-(i \bmod m)} b^{n-(i \bmod n)}$, $i \in \mathbb{N}$, must be transformed $\mathcal{C}$ has to process them appropriately. Since the integer i is not bounded, to this end $\mathcal{C}$ runs through cycles. But $\mathcal{C}$ cannot remember how often a cycle has been passed through. Therefore we observe that there exists a $j \in \mathbb{N}$ such that every transformation has emitted some output after consuming $(ab)^j$. In particular, this output must contain symbols a or symbols b but not at the same time.

Now consider inputs $(ab)^{jmn}a^p b^q$ and $(ab)^{jmn}a^{p'}b^{q'}$ where $0 \leq p, p' \leq m-1$ and $0 \leq q, q' \leq n-1$. There exist $m \cdot n$ such words which may have caused the emittance of symbols a or b. So we are concerned with $2 \cdot m \cdot n$ possibilities. Assume that transformations are in the same state for two of the possibilities. If $p \neq p'$ or $q \neq q'$ a contradiction follows since $(ab)^{jmn}a^p b^q a^{m-p'} b^{n-q'}$ must not be transformed but would be since $(ab)^{jmn}a^p b^q a^{m-p} b^{n-q}$ is. If $p = p'$ and $q = q'$, we have $(ab)^{jmn}a^p b^q = (ab)^{jmn}a^{p'}b^{q'}$. But there must exist two computations for this input which emit different symbols. So we obtain a contradiction since outputs with different symbols would be recorded.

So far $\mathcal{C}$ has at least $2 \cdot m \cdot n$ states. In addition, the transformation for the $2 \cdot m \cdot n$ possibilities cannot reach the initial state for $j \geq 1$. The initial state must be accepting and so the input could always be extended such that the transformation continues with emitting the wrong output symbols. Altogether it follows that $\mathcal{C}$ needs at least $2 \cdot m \cdot n + 1$ states.

Once we have found the tight bound of the intersection for the first components a natural question is to do the same for the second components:

The *intersection for the second components of* T_1 *and* T_2 is

$$T_1 \cap_2 T_2 = \{(u,v) \mid \exists\, u', u'' : (u',v) \in T_1, (u'',v) \in T_2, u \in \{u', u''\}\}$$

At a first glance the operation seems to be more expensive than $\cap_1$, but actually it makes no difference.

Theorem 5 *For any integers $m, n \in \mathbb{N}$ let $\mathcal{A}$ be an n-state and $\mathcal{B}$ be an m-state* RTD. *Then $2 \cdot m \cdot n + 1$ states are sufficient and necessary in the worst case for an* RTD *to realize the transduction $T(\mathcal{A}) \cap_2 T(\mathcal{B})$.*

Proof. For two arbitrary transductions T_1 and T_2 the theorem follows from Theorem 4, which gives the tight bound for $\cap_1$, and Theorem 2, which gives an upper bound of n for the inversion, by $T_1 \cap_2 T_2 = (T_1^{-1} \cap_1 T_2^{-1})^{-1}$.

3.4 Concatenation

Let T_1 and T_2 be two transductions realized by some RTDs. The *concatenation of* T_1 *and* T_2 is defined on element position as follows:

$$T_1 T_2 = \{(u_1 u_2, v_1 v_2) \mid (u_1, v_1) \in T_1, (u_2, v_2) \in T_2\}$$

Theorem 6 *For any integers $m, n \in \mathbb{N}$ let $\mathcal{A}$ be an m-state* RTD *and $\mathcal{B}$ be an n-state* RTD*. Then $m+n$ states are sufficient and necessary in the worst case for an* RTD *$\mathcal{C}$ to realize the transduction $T(\mathcal{A})T(\mathcal{B})$.*

Proof. The upper bound is due to the observation that in $\mathcal{C}$ one may connect the final states of $\mathcal{A}$ with the initial state of $\mathcal{B}$ via λ-transitions which emit the empty word.

Let $\mathcal{A} = \langle S_A, A_A, B_A, \delta_A, s_{0,A}, F_A\rangle$ and $\mathcal{B} = \langle S_B, A_B, B_B, \delta_B, s_{0,B}, F_B\rangle$ with $S_A \cap S_B = \emptyset$, then $\mathcal{C} = \langle S_A \cup S_B, A_A \cup A_B, B_A \cup B_B, \delta, s_{0,A}, F_B\rangle$ realizes the transduction $T(\mathcal{A})T(\mathcal{B})$, where

$$\delta(s,a) = \begin{cases} \delta_A(s,a) & \text{if } s \in S_A \setminus F_A \\ \delta_A(s,a) & \text{if } s \in F_A \text{ and } a \neq \lambda \\ \delta_B(s,a) & \text{if } s \in S_B \\ \delta_A(s,a) \cup \{(s_{0,B}, \lambda)\} & \text{if } s \in F_A \text{ and } a = \lambda \end{cases}$$

for $s \in S_A \cup S_B$ and $a \in A_A \cup A_B \cup \{\lambda\}$.

The upper bound is reached for the concatenation of the transductions $T(\mathcal{A}) = \{(a^{i\cdot m}, \$^i) \mid i \in \mathbb{N}_0\}$ and $T(\mathcal{B}) = \{(b^{i\cdot n}, \$^i) \mid i \in \mathbb{N}_0\}$. In [6] it has been shown that any NFA needs at least $m+n$ states to accept the language $\{a^m\}^*\{b^n\}^* = L(T(\mathcal{A})T(\mathcal{B}))$.

3.5 Reversal

Let T be a transduction realized by some RTD. The *reversal of T* is denoted by T^R and defined to be

$$T^R = \{(u,v) \mid (u^R, v^R) \in T\}$$

Theorem 7 *For any integer $n > 3$ let $\mathcal{A}$ be an n-state* RTD*. Then $n+1$ states are sufficient and necessary in the worst case for an* RTD *$\mathcal{C}$ to realize the transduction $T(\mathcal{A})^R$.*

Proof. Basically, the idea is to reverse the directions of the transitions and to interchange the meaning of the initial and accepting states. This works fine for RTDs whose set of final states is a singleton. In general we are concerned with more than one accepting state and have to add a new initial state as shown below. So we obtain an $(n+1)$-state RTD.

Let $\mathcal{A} = \langle S_A, A, B, \delta_A, s_{0,A}, F_A\rangle$ be an n-state RTD. Define $\mathcal{C} = \langle S, A, B, \delta, s_0, F\rangle$ according to $S = S_A \cup \{s_0\}$ where $s_0 \notin S_A$, $F = \{s_{0,A}\}$, and for $a \in A \cup \{\lambda\}$:

$$\delta(s,a) = \begin{cases} \{(s', b) \in S_A \times (B \cup \{\lambda\}) \mid (s,b) \in \delta_A(s', a)\} & \text{if } s \in S_A \\ \{(s', \lambda) \mid s' \in F_A\} & \text{if } s = s_0 \text{ and } a = \lambda \end{cases}$$

Clearly, the $(n+1)$-state RTD $\mathcal{C}$ realizes the reversal of $T(\mathcal{A})$.

The transduction

$$T_k = \{(a^{(i+1)\cdot k+i}b^j, \$^{i+j}) \mid i,j \in \mathbb{N}_0\} \cup \{(a^{(i+1)\cdot k+i}c^j, \$^{i+j}) \mid i,j \in \mathbb{N}_0\}$$

for $k \geq 1$ may serve as an example for the fact that the bound is reached. The $(k+3)$-state RTD which realizes T_k and the $(k+4)$-state RTD which realizes T_k^R are depicted in Figure 1.

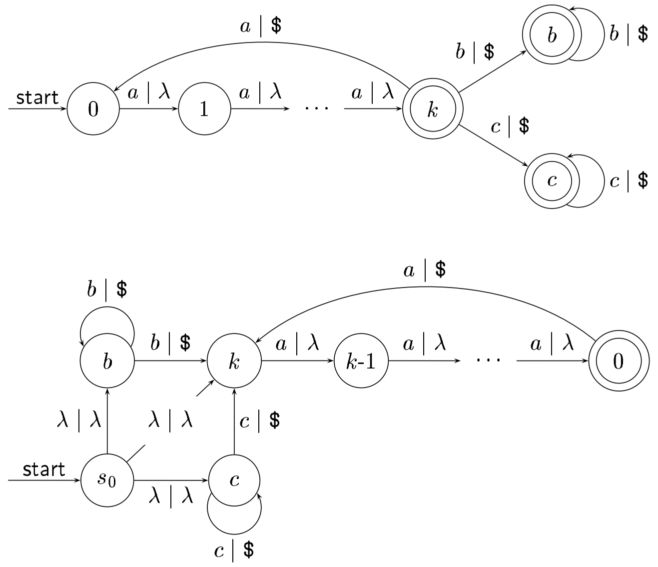

Figure 1: A $(k+3)$-state resp. a $(k+4)$-state RTD realizing T_k resp. T_k^R of Theorem 7.

The necessity of $k+4$ states follows once more from NFA acceptance. In [6] it has been shown that $k+4$ states are necessary to accept the language $L(T_k^R)$.

3.6 Homomorphism

Let $T \subseteq A^* \times B^*$ be a transduction realized by some RTD, C and D be two alphabets, and $g : A^* \to C^*$, $h : B^* \to D^*$ be two homomorphisms. The *homomorphic image of* T *under* g *and* h is defined on element position as follows:

$$T_{g,h} = \{(g(u), h(v)) \mid (u,v) \in T\}$$

Obviously, the number of states of an RTD realizing $T_{g,h}$ depends on T as well as on the homomorphisms g and h. In particular, we need to consider the structure of some minimal RTD $\mathcal{A} = \langle S, A, B, \delta, s_0, F\rangle$ which realizes T.

For easier writing let $R_{\mathcal{A}} \subseteq S \times (A \cup \{\lambda\}) \times S \times (B \cup \{\lambda\})$ be the set of possible transitions of $\mathcal{A}$, i.e., for $s, s' \in S$, $a \in A \cup \{\lambda\}$, and $b \in B \cup \{\lambda\}$:

$$(s,a,s',b) \in R_{\mathcal{A}} \iff (s',b) \in \delta(s,a)$$

For any $r = (s,a,s',b) \in R_{\mathcal{A}}$ define $x_r = \max\{|g(a)|, |h(b)|, 1\}$.

Theorem 8 *For any integer $n \in \mathbb{N}$ let $\mathcal{A} = \langle S, A, B, \delta, s_0, F\rangle$ be an n-state* RTD*, and $g : A^* \to C^*$ and $h : B^* \to D^*$ be two homomorphisms. Then*

$$n + \sum_{r \in R_{\mathcal{A}}} (x_r - 1)$$

states are sufficient for an RTD *$\mathcal{C}$ to realize the transduction $T_{g,h}(\mathcal{A})$.*

Proof. The underlying idea of construction is to replace each transition of $\mathcal{A}$ by a sequence of transitions during which the images of the input resp. output symbols are consumed resp. emitted. To this end define $\mathcal{C} = \langle S \cup S', C, D, \delta', s_0, F\rangle$ where S' and δ' are constructed as follows.

Initially let S' be empty and δ' be undefined for any arguments. The next step is performed for each $r = (s, a, s', b) \in R_{\mathcal{A}}$.

If $x_r = 1$, then define $(s, g(a), s', h(b))$ to be a possible transition in $\mathcal{C}$, i.e., let $(s', h(b)) \in \delta'(s, g(a))$. Otherwise we have $x_r > 1$. In this case proceed as follows: Join S' with $x_r - 1$ new states $s'_1, \dots, s'_{x_r-1}$. Build a word $c = c_1 \cdots c_{x_r}$ over C such that $c_1 \cdots c_{|g(a)|} = g(a)$ and $c_{|g(a)|+1}, \dots, c_{x_r} = \lambda$ if $|g(a)| < x_r$. Correspondingly, build a word $d_1 \cdots d_{x_r}$ over D with respect to $h(b)$. Now define the transition $(s'_1, d_1) \in \delta'(s, c_1)$ and set

$$\delta'(s'_1, c_2) = \{(s'_2, d_2)\},\ \delta'(s'_2, c_3) = \{(s'_3, d_3)\},\ \dots\ , \delta'(s'_{x_r-1}, c_{x_r}) = \{(s', d_{x_r})\}$$

The construction of $\mathcal{C}$ is complete when this procedure has been performed for any $r \in R_{\mathcal{A}}$. Since for any r exactly $x_r - 1$ new states are used we conclude $|S'| = \sum_{r \in R_{\mathcal{A}}} (x_r - 1)$ and, thus, the assertion of the theorem. The correctness of the construction can easily be shown by induction.

Now we turn to the question whether the number of states given in the previous theorem is necessary in the worst case. But what does this mean? It is easy to see that there exist transducers and homomorphisms such that the bound is matched. So the question could be whether it is possible to find such homomorphisms for any given (minimal) transducer. But even this answer is trivial since in fact there exists one pair of homomorphisms that is suitable for all (minimal) transducers. We simply may take the identity mappings.

The following example shows that there exist non-trivial homomorphisms for non-trivial RTDs.

Example 9 For any integer $n \in \mathbb{N}$ there exist an n-state RTD $\mathcal{A} = \langle \bar{S}, A, B, \bar{\delta}, \bar{s}_0, \bar{F}\rangle$ and homomorphisms $g : A^* \to \{a, b\}^*$, $h : B^* \to \{a, b\}^*$ such that any RTD $\mathcal{C}$ needs at least $|\bar{S}| + \sum_{r \in R_{\mathcal{A}}} (x_r - 1)$ states to realize the transduction $T_{g,h}(\mathcal{A})$.

In order to show the assertion consider the transduction

$$T = \{(u, v) \mid u \in \{a, b\}^*, \#_a(u) \equiv 0 \pmod{n}, v = \$^{\#_b(u)}\}$$

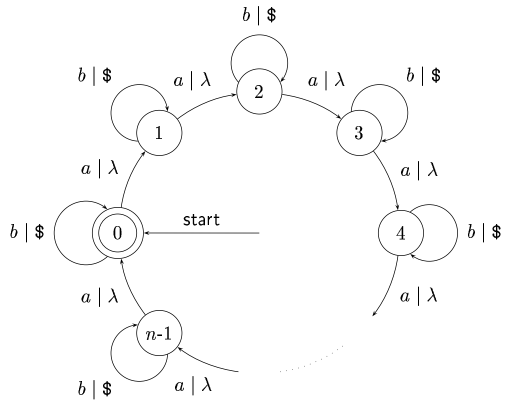

Figure 2: An n-state RTD realizing T of Example 9.

The n-state transducer $\mathcal{A}$ depicted in Figure 2 realizes T. For any $m \geq 1$ and $k \geq m$ consider the homomorphisms $g : \{a, b\}^* \to \{a, b\}^*$ where $g(a) = a^{m-1}b$, $g(b) = a^m$, and $h : \{\$\}^* \to \{a, b\}^*$ where $h(\$) = b^k$.

Let $\mathcal{C} = \langle S, \{a, b\}, \{a, b\}, \delta, s_0, F\rangle$ be an RTD that realizes $T_{g,h}$. We have to show that $\mathcal{C}$ has at least $n + n \cdot (m-1) + n \cdot (k-1) = n \cdot (m+k-1)$ states. To this end we analyze $\mathcal{C}$ for inputs of the form $g(ab^{i_1})g(ab^{i_2}) \cdots g(ab^{i_n})$ where $0 \leq i_1, \ldots, i_n$. For convenience we refer to the n subwords $g(a)$ by $w_1, \ldots, w_n$. So we have inputs $w_1 a^{i_1 \cdot m} w_2 \ldots w_n a^{i_n \cdot m}$ where $w_j = a^{m-1}b$, $1 \leq j \leq n$.

A transformation of such input runs through a sequence of configurations. We concentrate on configurations which are reached immediately by consuming an input symbol from the subwords w_j. Together with the initial configuration we obtain

$$\begin{aligned}
(s_0, w_1 a^{i_1 \cdot m} \cdots w_n a^{i_n \cdot m}, \lambda) &\vdash^+ (s_1, a^{m-2} b a^{i_1 \cdot m} \cdots w_n a^{i_n \cdot m}, v_1) \\
&\vdash^+ \cdots \vdash^+ (s_{m-1}, b a^{i_1 \cdot m} \cdots w_n a^{i_n \cdot m}, v_{m-1}) \\
&\vdash^+ (s_m, a^{i_1 \cdot m} \cdots w_n a^{i_n \cdot m}, v_m) \\
&\vdash^+ (s_{m+1}, a^{m-2} b a^{i_2 \cdot m} \cdots w_n a^{i_n \cdot m}, v_{m+1}) \\
&\vdash^+ \cdots \vdash^+ (s_{m \cdot n}, a^{i_n \cdot m}, v_{m \cdot n})
\end{aligned}$$

Obviously, all states $s_1, \ldots, s_{m \cdot n - 1}$ must be non-accepting. But, moreover, they must be distinct.

Contrarily, assume they are not, say $s_p = s_q$. Since $\mathcal{C}$ consumes at least one input symbol from the substrings w_j while getting from s_p to s_q we conclude that there exists a cycle which consumes more than one and not more than $m \cdot n - 1$ such input symbols. Therefore, there exist transformations which do not run resp. which run arbitrary times through the cycle . If the cycle consumes an input symbol b, a contradiction follows immediately since less than n symbols b are consumed. Otherwise the number of consumed symbols a must be a multiple of m where less than $m - 1$ and more than one are from the subwords w_j. In this case a contradiction follows when the transformation does not run through the cycle.

Consequently, the non-accepting states $s_1, \dots, s_{m \cdot n-1}$ must be distinct for any input of the required form. Together with the accepting initial state these are at least $m \cdot n$ states.

Next we consider the inputs $w_1 a^{i_1 \cdot m} w_2 w_3 \cdots w_n$, $0 \leq i_1$. Since i_1 may be arbitrary, $\mathcal{C}$ must have a cycle to process these inputs. If the cycle consumes at least one input symbol b, then the cycle length becomes arbitrarily long since all $i_1 \cdot m$ symbols a must have been consumed before. Therefore, we may assume that the cycle consumes only symbols a. Since a transformation may run through the cycle arbitrarily often when given the corresponding input, and $\mathcal{C}$ cannot remember how often, it follows that the $i_1 \cdot k$ output symbols b must be emitted in the cycle. This implies that the cycle length is at least k which is by definition not less than m. On the other hand, provided the corresponding input, a transformation may not run through the cycle at all. Thus, the sequence of states $s_0, \dots, s_{m \cdot n}$ mentioned above must be passed through apart from the cycle. By this observation it follows that the cycle may only start at one of the states s_m (reached by consuming the first b), $s_{m+1}, \dots, s_{2m-1}$, and afterwards the transformation must enter the successor state when consuming the next input symbol from the subword w_2. Due to the fact that a transition consuming an input symbol a may not nondeterministically emit b or λ at the same time, the cycle takes at least $k-1$ new states.

Now we consider the inputs $w_1 a^{i_1 \cdot m} w_2 a^{i_2 \cdot m} w_3 \cdots w_n$ and argue identically that another cycle starts at one of the states $s_{2m}, \dots, s_{3m-1}$ and afterwards the transformation must enter the successor state when consuming the next input symbol from the subword w_3.

By this properties it follows immediately that the cycles do not have common states. Repeating the argumentation we derive the necessity of n cycles in order to transform the inputs $w_1 a^{i_1 \cdot m} \cdots w_n a^{i_n \cdot m}$, $0 \leq i_1, \dots, i_n$. So the total number of necessary states is $n \cdot m + n \cdot (k-1) = n(m+k-1)$ what completes the analysis.

3.7 Composition

Let $T_1 \subseteq A_1^* \times B_1^*$ and $T_2 \subseteq A_2^* \times B_2^*$ be two transductions realized by some RTDs such that $B_2 \subseteq A_1$. Then

$$T_1 \circ T_2 = \{(u, v) \mid \exists\, w : (u, w) \in T_2 \wedge (w, v) \in T_1\}$$

is the *composition of* T_1 *and* T_2.

Theorem 10 *For any integers $m, n \in \mathbb{N}$ let $\mathcal{A}$ be an m-state* RTD *and $\mathcal{B}$ be an n-state* RTD *such that the set of output symbols of $\mathcal{B}$ is included in the set of input symbols of $\mathcal{A}$. Then $m \cdot n$ states are sufficient and necessary in the worst case for an* RTD *$\mathcal{C}$ to realize the transduction $T(\mathcal{A}) \circ T(\mathcal{B})$.*

Proof. The upper bound of $m \cdot n$ states is given by the cross-product construction of $\mathcal{C}$. Let $\mathcal{A} = \langle S_A, A_A, B_A, \delta_A, s_{0,A}, F_A \rangle$ and $\mathcal{B} = \langle S_B, A_B, B_B, \delta_B, s_{0,B}, F_B \rangle$ with $B_B \subseteq A_A$. Then $\mathcal{C} = \langle S_A \times S_B, A_B, B_A, \delta, (s_{0,A}, s_{0,B}), F_A \times F_B \rangle$ realizes the transduction $T(\mathcal{A}) \circ T(\mathcal{B})$ where

$$\delta((s_1, s_2), a) = \{((s_1', s_2'), b) \mid \exists\, c : \ (s_2', c) \in \delta_B(s_2, a), (s_1', b) \in \delta_A(s_1, c)\}$$

for $(s_1, s_2) \in S_A \times S_B$ and $a \in A_B \cup \{\lambda\}$.

As witnesses for the fact that the bound is reached in the worst case define for $r \in \mathbb{N}$ alphabets $A_r = \{a_1, \dots, a_r\}$, for $1 \le p \le r$ homomorphisms $h_p : A_r^* \to A_r^*$ where $h_p(a) = \lambda$ if $a = a_p$ and $h_p(a) = a$ otherwise, and transductions

$$T_{p,k} = \{(u, v) \mid u \in A_r^*, \#_{a_p}(u) \equiv 0 \pmod{k}, v = h_p(u)\}$$

for all $k \in \mathbb{N}$.

An RTD which realizes $T_{1,k}$ with k states is depicted in Figure 3.

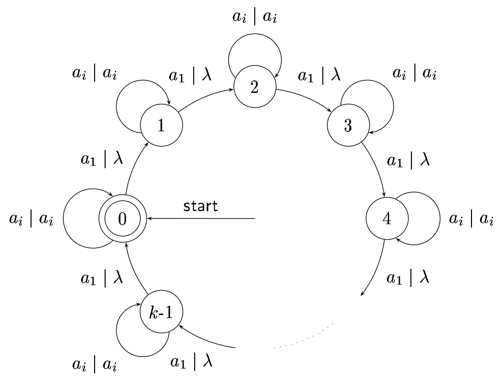

Figure 3: A k-state RTD realizing $T_{1,k}$ of Theorem 10 ($i \in \{2, \dots, r\}$).

It remains to show that an RTD $\mathcal{C}$ for $T_{p,m} \circ T_{q,n}$, $p \neq q$ and $m, n \in \mathbb{N}$ needs at least $m \cdot n$ states. Consider the inputs $a_p^i a_q^j a_o^\ell$ and $a_p^{i'} a_q^{j'} a_o^\ell$ with $0 \le i, i' \le m-1$, $0 \le j, j' \le n-1$, an arbitrary but fixed $\ell \in \mathbb{N}$, and $o \neq p$, $o \neq q$. Let $\mathcal{C}$ be in configurations $c_{ij} = (s_{ij}, \lambda, v_{ij})$ resp. $c_{i'j'} = (s_{i'j'}, \lambda, v_{i'j'})$ after processing the corresponding inputs. Since $(a_p^i a_q^j a_o^\ell a_p^{m-i} a_q^{n-j}, a_o^\ell)$ and $(a_p^{i'} a_q^{j'} a_o^\ell a_p^{m-i'} a_q^{n-j'}, a_o^\ell)$ belong to $T_{p,m} \circ T_{q,n}$ there must exist such configurations from which final configurations are reachable when the input is extended appropriately:

$$(s_{ij}, a_p^{m-i} a_q^{n-j}, v_{ij}) \vdash^* (s_f, \lambda, a_o^\ell) \quad \text{and} \quad (s_{i'j'}, a_p^{m-i'} a_q^{n-j'}, v_{i'j'}) \vdash^* (s_f', \lambda, a_o^\ell)$$

for some final states s_f, s_f'.

Assume $\mathcal{C}$ has less than $m \cdot n$ states. There exist $m \cdot n$ different inputs of the form in question. Therefore at least for two different inputs we have $s_{ij} = s_{i'j'}$. This implies $(s_0, a_p^{i'} a_q^{j'} a_o^\ell, \lambda) \vdash^* (s_{i'j'}, \lambda, v_{i'j'}) = (s_{ij}, \lambda, v_{i'j'})$ and further $(s_{ij}, a_p^{m-i} a_q^{n-j}, v_{i'j'}) \vdash^* (s_f, \lambda, v_{i'j'} w)$. Thus $(a_p^{i'} a_q^{j'} a_o^\ell a_p^{m-i} a_q^{n-j}, v_{i'j'} w)$ must belong to $T_{p,m} \circ T_{q,n}$. But contrarily, either $i \neq i'$ or $j \neq j'$, and thus $i' + m - i \not\equiv 0 \pmod{m}$ or $j' + n - j \not\equiv 0 \pmod{n}$, a contradiction.

The transductions $T_{p,k}$ are realizable by deterministic rational transducers. Therefore the tight bound for composition holds also in the deterministic case. This is of particular interest since the sole positive closure (under the operations in question) of deterministic transducers is under composition, and even for this operation there is no difference in the state complexities.

Finally, Table 1 summarizes the shown state complexity bounds for RTDs.

	RTD
$\cup$	$m+n+1$
$\cap_1$	$2 \cdot m \cdot n + 1$
$\cap_2$	$2 \cdot m \cdot n + 1$
$\cdot$	$m+n$
R	$n+1$
T^{-1}	n
$\circ$	$m \cdot n$
h	$n + \sum_{r \in R}(x_r - 1)$

Table 1: State complexities of basic operations on rational transductions. For r, R and x_r see Section 3.6.

References

[1] Aho, A., Sethi, R., and Ullman, J. D. *Compilers: Principles and Techniques and Tools*. Addison-Wesley, Reading, 1986.

[2] Berstel, J. *Transductions and Context-Free-Languages*. Teubner, Stuttgart, 1979.

[3] Choffrut, C. and Čulik II, K. *Properties of finite and pushdown transducers*. SIAM Journal on Computing 12 (1983), 300–315.

[4] Čulik II, K. and Karhumäki, J. *The equivalence problem for single-valued two-way transducers (on NPDTOL languages) is decidable*. SIAM Journal on Computing 16 (1987), 221–230.

[5] Eilenberg, S. *Automata, Languages, and Machines*. Academic Press, New York, 1974.

[6] Holzer, M. and Kutrib, M. *State complexity of basic operations on nondeterministic finite automata*. In *Implementation and Application of Automata (CIAA 2002)*.

[7] Ibarra, O. H. *The unsolvability of the equivalence problem for ε-free NGSM's with unary input (output) alphabet and applications*. SIAM Journal on Computing 7 (1978), 524–532.

[8] Mohri, M. *Finite-state transducers in language and speech processing*. Computational Linguistics 23 (1997), 269–311.

[9] Mohri, M. *Minimization algorithms for sequential transducers*. Theoretical Computer Science 234 (2000), 177–201.

[10] Nerode, A. and Kohn, W. *Models for hybrid systems: Automata, topologies, controllability, observability*. Hybrid Systems, LNCS 736, 1993, pp. 317–356.

[11] Schützenberger, M. P. *Sur les relations rationelles*. Automata Theory and Formal Languages, LNCS 33, 1975, pp. 209–213.

[12] Weber, A. *Transforming a single-valued transducer into a Mealy machine*. Journal of Computer and System Sciences 56 (1998), 46–59.

[13] Weber, A. and Klemm, R. *Economy of description for single-valued transducers*. Information and Computation 118 (1995), 327–340.

[14] Yu, S. *Regular languages*. In Rozenberg, G. and Salomaa, A. (eds.), *Handbook of Formal Languages 1*. Springer, Berlin, 1997, chapter 2, pp. 41–110.

[15] Yu, S. *State complexity of regular languages*. Journal of Automata, Languages and Combinatorics 6 (2001), 221–234.